PROBLEM SET X
(due September 1 2020)

Problem 1

Three charges are situated at the corners of a square of side $a$ as shown in the figure below. How much work does it take to bring another charge $+q$ from far away and place it in the fourth corner?

![Diagram of three charges in a square]

Problem 2

Refer to the system previously discussed in Problem 1. How much work does it take to assemble the whole configuration of four charges?
Problem 3

Consider an infinite one-dimensional chain of point charges $q$ and $-q$ with alternating signs strung out along the x-axis each with a distance $a$ from its nearest neighbor. Compute the work per particle to assemble this system. As a hint consider the Maclaurin series expansion of

$$\ln(1 + x)$$

Problem 4

Two positive charges point charges $q_a$ and $q_b$ of masses $m_a$ and $m_b$ respectively are at rest held together by a massless string of length $a$. If the string is cut, the two particles fly off in opposite directions. How fast is each one going when they are far apart?

Problem 5

Show that the electrostatic potential energy $U$ of an arrangement of eight separate negative charges $-q$ on the corners of a cube of side $b$ with a positive charge $+2q$ located at the center of the cube.
is given by

\[ U = \frac{1}{4\pi \varepsilon_0} \left[ \frac{-16 q^2}{\sqrt{3} b} + \frac{12 q^2}{b} + \frac{12 q^2}{\sqrt{2} b} + \frac{4 q^2}{\sqrt{3} b} \right] = \frac{q^2}{4 b \pi \varepsilon_0} \cdot 4.31947 \]

You can use as a hint the figure below where you should recall that we only have to worry about calculating pairs of interactions among charges when evaluating the electrostatic potential energy of a system.
Problem 6

Find the capacitance of two coaxial spherical metal shells with radii $r_a$ and $r_b$ where $r_a < r_b$. Note that the inner sphere has a charge $+q$ while the outer sphere has a charge $-q$.

Problem 7

Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii $r_a$ and $r_b$ where $r_a < r_b$ as shown in the figure below. Note that the inner cylinder has a surface charge density $+\sigma$ while the outer cylinder has a surface charge density $-\sigma$.

Problem 8

Consider the parallel-plate capacitor shown in the figure below
If you wish to "charge up" the capacitor, you have to remove electrons from the positive plate and move them to the negative plate. In doing so, you fight against the electric field which is pulling them back towards the positive plate and pushing them away from the negative one. How much work is then done to charge the capacitor up to some final amount $Q$? We can calculate this by supposing that at some intermediate stage the charge on the positive plate is $q$. This would mean that the electrostatic potential difference between the two plates of the capacitor is $\frac{q}{C}$. The work required to move the next piece of charge $dq$ is therefore

$$dW = \frac{q}{C} dq$$

Show that the total work required to go from $q = 0$ to $q = Q$ is

$$W = \frac{1}{2} CV^2$$

where $V$ is the final electrostatic potential of the capacitor.
Problem 1

Three charges are situated at the corners of a square of side \( a \), as shown in the figure below. How much work does it take to bring in another charge \( q \) from far away and place it in the fourth corner?

![Diagram of a square with charges at the corners]

From Lecture 10 we know the extra work required to bring in \( q \) to our system of charges is

\[
W_q = \frac{1}{4\pi\varepsilon_0} q \left[ \frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right]
\]
Using our figure we see

\[ g_1 = -g \quad \Rightarrow r_{1y} = a \]
\[ g_2 = +g \quad \Rightarrow r_{2y} = \sqrt{2}a \]
\[ g_3 = -g \quad \Rightarrow r_{3y} = a \]
\[ g_4 = +g \]

\[ W_y = \frac{1}{4\pi\varepsilon_0} g \left[ \frac{-g}{a} + \frac{g}{\sqrt{2}a} + \frac{-g}{a} \right] \]
\[ W_y = \frac{g^2}{4\pi\varepsilon_0} \left[ -1 + \frac{1}{\sqrt{2}} - 1 \right] \]
\[ W_y = \frac{g^2}{4\pi\varepsilon_0} \left[ \frac{1}{\sqrt{2}} - 2 \right] \]

\( W_y < 0 \) so system is stable
Problem 2

Refer to the system described in Problem 1. How much work does it take to assemble the whole configuration of four charges?

\[
W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{4} \sum_{j=1, j \neq i}^{4} \frac{q_i q_j}{r_{ij}}
\]

\[
= \frac{1}{8\pi\varepsilon_0} \left[ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_2 q_1}{r_{21}} + \frac{q_3 q_2}{r_{32}} + \frac{q_3 q_4}{r_{34}} + \frac{q_4 q_1}{r_{41}} + \frac{q_4 q_2}{r_{42}} + \frac{q_4 q_3}{r_{43}} \right]
\]
From our diagram

\[ q_1 = -q \quad r_{12} = a \]
\[ q_2 = -q \quad r_{13} = \sqrt{2} a \]
\[ q_3 = -q \quad r_{14} = a \]
\[ q_4 = +q \quad r_{21} = a \]

\[ r_{31} = \sqrt{2} a \quad r_{23} = a \]
\[ r_{32} = a \quad r_{24} = \sqrt{2} a \]
\[ r_{34} = a \quad r_{41} = a \]
\[ r_{42} = \sqrt{2} a \quad r_{43} = a \]

\[ W = \frac{1}{8 \pi \sigma_0} \left[ \frac{-q^2}{a} + \frac{q^2}{\sqrt{2} a} - \frac{-q^2}{a} \right] \]

\[ W = \frac{1}{8 \pi \sigma_0} \left[ -\frac{q^2}{a} + \frac{\sqrt{2} q^2}{a} \right] \]

\[ W = \frac{q^2}{2 \pi \sigma_0 a} \left[ \frac{1}{\sqrt{2}} - 2 \right] < 0 \quad \text{System is stable!} \]
Problem 3

Consider an infinite one-dimensional chain of point charges $q$ and $-q$ with alternating signs strung out along the x-axis each with a distance $a$ from its nearest neighbor. Compute the work per particle to assemble this system. As a hint consider the Maclaurin series expansion of

$$\ln (1+x)$$

This is a classic problem but it needs to be approached carefully as it has some subtle points that should not be overlooked.

Let us start with

$$(3-1) \quad W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{r_{ij}}$$

which is the work required to assemble a system of $N$ charges. In our problem we do not have $N$ charges but an infinite number. Let us pick one charge $q_i$ in $(3-1)$ and construct our system accordingly.
If we select \( g_i \) then (3-1) can be expanded as

\[
(3-2) \quad W = \frac{1}{8\pi G_0} \left[ \frac{g_i g_{i+1}}{a} + \frac{g_i g_{i-1}}{a} + \frac{g_i g_{i+2}}{2a} + \frac{g_i g_{i-2}}{2a} + \frac{g_i g_{i+3}}{3a} + \frac{g_i g_{i-3}}{3a} + \ldots \right]
\]

Since \( g_i = +g \) by choice but all of the following still holds if we chose \( g_i = -g \)

\( g_{i+1} = -g \)

\( g_{i-1} = -g \)

\( g_{i+2} = +g \), etc

and (3-2) becomes

\[
(3-3) \quad W = \frac{1}{8\pi G_0} \left[ -\frac{g^2}{a} - \frac{g^2}{a} + \frac{g^2}{2a} + \frac{g^2}{2a} - \frac{g^2}{3a} - \frac{g^2}{2a} + \ldots \right]
\]

or

\[
(3-4) \quad W = \frac{g^2}{4\pi G_0 a} \left[ -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots \right]
\]
By factoring out \(-1\) we get

\[
W = \frac{g^2}{4\pi^2} \left[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \right]
\]

The terms in the bracket can be summed exactly if we use our hint and do a Maclaurin series expansion for \(\ln(1+x)\)

\[
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}
\]

This series converges for \(1 < x < 1\), but we need to explore what happens at the radii of convergence, namely \(x = 1\) and \(x = -1\).

For \(x = -1\) this series becomes the harmonic series which diverges

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{2n} \frac{(-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}
\]
For \( x = 1 \) we find

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1)^n}{n} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n}
\]

which is the alternating harmonic series which converges conditionally.

Thus

\[
\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}
\]

converges for \( x \) in the interval \(-1 < x \leq 1\) and

\[
W = \frac{-g^2}{4\pi\epsilon_0 a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1)^n}{n} = \frac{-g^2}{4\pi\epsilon_0 a} \ln(1+1) = \frac{-g^2}{4\pi\epsilon_0 a} \ln(2)
\]

or

\[
W = \frac{-g^2}{4\pi\epsilon_0 a} \ln 2
\]

Note that this is the work per particle required to assemble the infinite system of point charges in one-dimension.
Problem 41

Two positive charges (point charges) of charges $q_1$ and $q_2$ of masses $m_1$ and $m_2$ respectively are at rest held together by a massless string of length $a$. If the string is cut, the two particles fly off in opposite directions. How fast is each one going when they are far apart?

\[ q_1 \rightarrow a \rightarrow q_2 \]

Our system initially has no kinetic energy but only electrostatic potential energy $U$

\[ U = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{a} \quad (4-1) \]

When the string is cut the two particles fly off in opposite directions due to Coulomb's Law. At some far apart distance all of the initial electrostatic potential energy of the system is converted to the kinetic energy of both particles, at which point $V_A$ and $V_B$ are constants.

\[ \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{a} = \frac{m_1 V_A^2}{2} + \frac{m_2 V_B^2}{2} \quad (4-2) \]
We need to find $V_A, V_B$ but we lack a second equation. Conservation of momentum comes to our rescue.

\[(4-3) \quad m_A V_A = m_B V_B\]

Where $V_A$ and $V_B$ are the velocities of the two charges when they are far apart.

If we combine (4-2) and (4-3) we discover:

\[
\frac{1}{4\pi\varepsilon_0} \frac{g_A g_B}{a} = \frac{m_A}{2} V_A^2 + \frac{m_B}{2} \left(\frac{m_A}{m_B}\right)^2 V_B^2
\]

or

\[
V_B^2 \left[ \frac{m_A}{2} + \frac{m_A^2}{2m_B} \right] = \frac{g_A g_B}{4\pi\varepsilon_0 a}
\]

or

\[
V_A^2 \left[ \frac{m_A m_B}{2m_B} + \frac{m_A^2}{2m_B} \right] = \frac{g_A g_B}{4\pi\varepsilon_0 a}
\]

\[
V_A^2 \left[ \left( \frac{m_A}{m_B} \right) \left[ \frac{m_B}{2} + \frac{m_A}{2} \right] \right] = \frac{g_A g_B}{4\pi\varepsilon_0 a}
\]

\[
V_A = \left( \frac{g_A g_B}{4\pi\varepsilon_0 a} \right)^{1/2} \left[ \left( \frac{m_A}{m_B} \right) \left[ \frac{m_A + m_B}{2} \right] \right]^{-1/2}
\]
\[ V_A = \left( \frac{g_A g_B}{4\pi \varepsilon_0 c} \right)^{\frac{1}{2}} \left[ \frac{m_B}{m_A} \right] \left[ \frac{m_B + m_B}{2} \right]^{\frac{1}{2}} \]

\[ V_B = \left( \frac{g_A g_B}{4\pi \varepsilon_0 c} \right)^{\frac{1}{2}} \left[ \frac{m_B}{m_A} \right] \left[ \frac{m_B + m_B}{2} \right]^{\frac{1}{2}} \]

Similarly,

\[ V_B = \left( \frac{g_B g_B}{2\pi \varepsilon_0 c} \right)^{\frac{1}{2}} \left[ \frac{m_B}{m_B} \right] \left[ \frac{1}{m_B + m_B} \right] \left( (4-4) \right) \]

\[ V_B = \left( \frac{g_B g_B}{2\pi \varepsilon_0 c} \right)^{\frac{1}{2}} \left[ \frac{m_B}{m_B} \right] \left[ \frac{1}{m_B + m_B} \right] \left( (4.5) \right) \]
Problem 5

Show that the electrostatic potential energy $U$ of an arrangement of eight separate negative charges $-q$ on the corners of a cube of side $b$ with a positive charge of $+2q$ located at the center of the cube
is given by

\[ U = \frac{1}{4 \pi \varepsilon_0} \left[ \frac{-16 b^2}{\sqrt{5} b} + \frac{12 g^2}{b} + \frac{12 g^2}{\sqrt{2} b} + \frac{4 g^2}{\sqrt{3} b} \right] \]

or

\[ U = \frac{g^2}{4 \pi \varepsilon_0} = 4.31947 \ldots \]

You can use as a hint the figure below where you should recall we only need worry about calculating pairs of interactions among charges when evaluating the electrostatic potential energy of a system.

12 such pairs

4 such pairs

8 such pairs
Given
\[ U = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j=i}^{N} \frac{q_i q_j}{r_{ij}} \]

we only need consider pairs of interactions among charges once so we don't need our factor of \( \frac{1}{2} \)

\[ U = \frac{1}{8\pi\varepsilon_0} \left( \# \text{ pairs of type 1} + \# \text{ pairs of type 2} \right) + \ldots + 7 \]

For our cube:

- Face diagonal is \( \sqrt{2} b \)
  since \( b^2 + b^2 = 2b^2 \)

(b) Cube diagonal

\[ a^2 = 2b^2 + b^2 \]
\[ a = \sqrt{3} b \]
(c) Half of cube diagonal

\[
\frac{\sqrt{3} b}{2} \quad \frac{\sqrt{2} b}{\sqrt{2}}
\]

From our figure:

There are 12 pairs of interactions between \(-g\) and \(-g\) of separation \(b\).

There are six cube faces and 12 pairs of interactions between \(-g\) and \(-g\) considering face diagonal interactions of distance \(\sqrt{2} b\).

There are four pairs of interactions between \(-g\) and \(-g\) considering body diagonal interactions of distance \(\sqrt{3} b\).

There are eight pairs of interactions between \(-g\) and \(+2g\) considering half the distance \(\frac{\sqrt{3}b}{2}\) for body diagonal interactions.
If we put all of this together, we obtain

\[ U = \frac{1}{4 \pi \varepsilon_0} \left[ \frac{8}{\sqrt[3]{b}} (-2a^2) + \frac{12 a^2}{b} + \frac{12 a^2}{\sqrt{3} b} + \frac{4 a^2}{\sqrt{3} b} \right] \]

\[ U = \frac{q^2}{4 \pi \varepsilon_0} \left( 4.31947 \right) > 0 \]

You have to do work to assemble this system as it is unstable and it will fly apart left to its own devices (sans external force)!
Problem 6

Find the capacitance of two coaxial spherical metal shells with radii $r_a$ and $r_b$, where $r_a < r_b$. Note that the inner sphere has a charge $+q$ while the outer sphere has a charge $-q$.

Use Gauss's Law in each of the three regions.

Region I: $0 < r < r_a$

Gaussian surface (sphere)

\[
\oint E \cdot dS = \frac{q_{enc}}{\varepsilon_0} = \oint E \cdot dS \quad \text{by symmetry}
\]

Thus $E = 0$ and $E = \frac{q_{enc}}{\varepsilon_0} = E \int_{S} dS$.
Region II \( (T_2 < T < T_b) \)

Gaussian surface (sphere)

Using the same arguments as in Region II

\[
\oint \mathbf{E} \cdot d\mathbf{s} = E 4\pi r^2 = \frac{\rho r^2}{\varepsilon_0} = \frac{1}{\varepsilon_0} + q
\]

\[
\mathbf{E} = \frac{q}{4\pi \varepsilon_0 r^2}
\]

Region III \( (T_b < T < \infty) \)

Gaussian surface (sphere)

Using the same arguments as in Regions I & II

\[
\oint \mathbf{E} \cdot d\mathbf{s} = E 4\pi r^2 = \frac{\rho r^2}{\varepsilon_0} = \frac{[q - \delta]}{\varepsilon_0} = 0
\]

\[
\mathbf{E} = 0
\]
There is only an $E$ field in Region II.

\[
V = -\int_{r_0}^{r_2} E \cdot dr = \int_{r_0}^{r_2} E \cdot \hat{r} \, dr
\]

\[
V = -\int \frac{\mathcal{E}}{4\pi\varepsilon_0} \, dr
\]

\[
V = \frac{\mathcal{E}}{4\pi\varepsilon_0} \left[ \frac{1}{r_2} - \frac{1}{r_3} \right] > 0
\]

since $r_3 > r_2$

\[
C = \frac{1}{V} = \frac{1}{V} \int \frac{3}{4\pi\varepsilon_0} \left( \frac{r_3 - r_2}{r_2 r_3} \right) \quad \text{d}r = \frac{4\pi\varepsilon_0}{r_3 - r_2} \frac{r_2 r_3}{r_3 - r_2} > 0
\]

This is entirely a geometric factor!
Problem 7

Find the capacitance per unit length of two coaxial metal tubes of radii \( r_a \) and \( r_b \) where \( r_a < r_b \) as shown in the figure below.

Note that the inner cylinder has a surface charge density \( +\sigma \) while the outer cylinder has a surface charge density \( -\sigma \).

Use Gauss's Law in each of the three regions.

Region I \( 0 < r < r_a \)

Break Gaussian surface up into 3 sections:

\[
\oint E \cdot d\mathbf{s} = \int \frac{\rho}{\varepsilon_0} d\mathbf{s}
\]

Gaussian surface cylinder of length \( L \)

Top

Bottom

Side
By symmetry \[ \mathbf{E} \cdot d\mathbf{s} \] along side
and \[ \mathbf{E} \cdot d\mathbf{s} \] on top and bottom
so only surface integral over side
of Gaussian cylinder survives

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \int \mathbf{E} \cdot d\mathbf{s} = \mathbf{E} \int d\mathbf{s} = \mathbf{E} 2\pi r L \]

\[ \mathbf{E} \text{ is constant} \]
over \( S \) by symmetry

\[ \frac{\varepsilon_0 \mathbf{E}}{\varepsilon_0} = 0 \]

\[ \mathbf{E} = 0 \]

Region II \[ \tau_a < r < \tau_b \]

Using the same arguments as discussed in Region I

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \mathbf{E} 2\pi r L = \varepsilon_0 \mathbf{E}_0 = \varepsilon_0 \frac{\mathbf{E}}{\mathbf{E}_0} = \varepsilon_0 \frac{\Delta \mathbf{A}}{\mathbf{E}_0} = \varepsilon_0 \frac{2\pi r L}{\mathbf{E}_0} \]

Gaussian
surface
(cylinder)

\[ \mathbf{E} = \frac{\sigma \mathbf{r}_a}{\mathbf{E}_0} \hat{\mathbf{r}} \]

\[ \mathbf{E} = \frac{\sigma \mathbf{r}_a}{\mathbf{E}_0} \hat{\mathbf{r}} \]
Region III \( r_b < r < \infty \)

Using the same arguments as discussed in Region I and Region II

\[
\int \vec{E} \cdot d\vec{s} = E 2\pi r = \frac{\sigma A}{\varepsilon_0} = \frac{\sigma - \sigma}{\varepsilon_0} = 0
\]

\[E = 0\]

To find the capacitance we must compute the potential difference

\[
V = -\int \vec{E} \cdot d\vec{r} = -\int \frac{\sigma}{\varepsilon_0} \frac{r_b}{r} dr - \int \frac{\sigma}{\varepsilon_0} \frac{r_a}{r} dr
\]

\[
= -\int_{r_b}^{r_a} \frac{\sigma}{\varepsilon_0} \frac{r_a}{r} dr = -\sigma r_a \ln \left( \frac{r_a}{r_b} \right)
\]

\[V = \frac{\sigma}{\varepsilon_0} r_a \ln \left( \frac{r_a}{r_b} \right) > 0\]
\[
C = \frac{q}{V} = \frac{\varepsilon_0}{\sigma \tau_a \ln \left( \frac{\tau_b}{\tau_a} \right)}
\]

\[
C = \frac{\varepsilon_0}{\tau_a} \frac{1}{\ln \left( \frac{\tau_b}{\tau_a} \right) \left( \frac{a}{l} \right) ^{-1}}
\]

\[
C = \frac{\varepsilon_0}{\tau_a} \frac{2 \pi \tau a l}{2} \frac{1}{\ln \left( \frac{\tau_b}{\tau_a} \right)}
\]

\[
C = \frac{2 \pi \varepsilon_0 L}{\ln \left( \frac{\tau_b}{\tau_a} \right)}
\]

\[
\frac{C}{L} = \frac{2 \pi \varepsilon_0}{\ln \left( \frac{\tau_b}{\tau_a} \right)}
\]

Capacitance per unit length
Problem 8

Consider the parallel-plate capacitor shown in the figure below.

![Diagram of a parallel-plate capacitor]

If you wish to "charge up" the capacitor, you have to remove electrons from the positive plate and move them to the negative plate. In doing so, you fight against the electric field which is pulling them back towards the positive plate and pushing them away from the negative one. How much work is then done to charge the capacitor up to some final amount \( Q \)? We can calculate this by supposing that at some intermediate state the charge on the positive plate is \( Q \). This would mean that the electrostatic potential difference between the two plates of the capacitor is

\[
V = \frac{Q}{C}
\]
The work required to move the next piece of charge $dq$ is therefore

$$dW = Vdq = \frac{q}{C} dq$$

Show that the total work required to go from $q=0$ to $q=a$ is

$$W = \frac{1}{2} CV^2$$

$$W = \int_{0}^{a} dW = \int_{0}^{a} Vdq = \int_{0}^{a} \frac{q}{C} dq = \frac{a^2}{2C}$$

Since $V$ is the final electrostatic potential of the capacitor

$$C = \frac{Q}{V}$$

and

$$W = \frac{a^2}{2C} = \frac{C^2 V^2}{2C} = \frac{1}{2} CV^2$$

$$W = \frac{1}{2} CV^2$$