\[ \sin \theta' d\theta' = \frac{du}{2 r' z} \]

and Eq. (1) becomes

\[ V(z) = \frac{\rho}{2 \epsilon_o} \int \int \frac{1}{2 z \sqrt{r'}} \frac{du}{\sqrt{u}} (r')^2 dr' \]

Now integrate the inner integral over \( u \) and use Eq. (2) to find the appropriate limits of integration as the polar angle goes from \( \theta' = 0 \) to \( \theta' = \pi \). You should obtain the result

\[ V(z) = \left[ \frac{\rho}{2 \epsilon_o z} \int \sqrt{(z^2 + r')^2 - \sqrt{(z^2 - r')^2}} \right] \frac{r'}{dr'} \]

To evaluate this integral only take the positive square roots and be sure that you are outside the sphere. You should get a familiar result for the electrostatic potential \( V \). Use this result to get a familiar result for the electric field \( \vec{E} \)

\[ \vec{E} = \int \frac{q}{4\pi \epsilon_o r^2} \hat{r} \]

where \( q \) is the total charge of the sphere. Note that your answer is identical to the case where the total charge \( q \) is concentrated at the center of the sphere which is neat!

**Problem 2**

In Lecture 9 we calculated the electrostatic potential \( \text{inside} \) a uniformly charged solid sphere of radius \( a \) and total charge density \( q \). We left out a few steps in our derivation and left them for this problem set.

Let us start with the step where we have done the azimuthal integral

\[ V(z) = \frac{\rho}{2 \epsilon_o} \int \int \frac{(r')^2 dr' \sin \theta' d\theta'}{\sqrt{z^2 + (r')^2 - 2 z r' \cos \theta'}} \]

Next we must perform the polar integral over \( \theta' \). We first introduce a dummy variable \( u \) where

\[ u = z^2 + (r')^2 - 2 z r' \cos \theta' \]

When we integrate over the polar angle \( \theta' \), we realize from our original figure that \( r' \) is fixed. Note also that we treat \( z \) as a constant when we do the integral over \( \theta' \) since we are evaluating the electrostatic potential \( V(z) \) at a fixed value of \( z \). Given these observations show that
\[ \sin \theta' \, d\theta' = \frac{du}{2 \, r' \, z} \]

and Eq. (1) becomes

\[ V(z) = \frac{\rho}{2 \epsilon_o} \int \int \frac{1}{2 \, z \, r'} \frac{du}{\sqrt{u}} \left( r' \right)^2 \, dr' \]

Now integrate the inner integral over \( u \) and use Eq. (2) to find the appropriate limits of integration as the polar angle goes from \( \theta' = 0 \) to \( \theta' = \pi \). You should obtain the result

\[ V(z) = \left[ \frac{\rho}{2 \epsilon_o \, z} \int \sqrt{\left( z^2 + r' \right)^2 - \sqrt{\left( z^2 - r' \right)^2}} \right] r' \, dr' \]

To evaluate this integral only take the positive square roots and be sure that you are outside the sphere. You should get a familiar result for the electrostatic potential \( V \). Use this result to get a familiar result for the electric field \( \vec{E} \)

\[ \vec{E} = \int \frac{q}{4\pi \epsilon_o r^2} \hat{r} \]

where \( q \) is the total charge of the sphere. Note that your answer is identical to the case where the total charge \( q \) is concentrated at the center of the sphere which is neat!

**Problem 2**

In Lecture 9 we calculated the electrostatic potential *inside* a uniformly charged solid sphere of radius \( a \) and total charge density \( q \). We left out a few steps in our derivation and left them for this problem set.

Let us start with the step where we have done the azimuthal integral

\[ V(z) = \frac{\rho}{2 \epsilon_o} \int \int \frac{(r')^2 \, dr' \, \sin \theta' \, d\theta'}{\sqrt{z^2 + (r')^2 - 2 \, z \, r' \, \cos \theta'}} \]

Next we must perform the polar integral over \( \theta' \). We first introduce a dummy variable \( u \) where

\[ u = z^2 + (r')^2 - 2 \, z \, r' \, \cos \theta' \]

When we integrate over the polar angle \( \theta' \) we realize from our original figure that \( r' \) is fixed. Note also that we treat \( z \) as a constant when we do the integral over \( \theta' \) since we are evaluating the electrostatic potential \( V(z) \) at a fixed value of \( z \). Given these observations show that
\[ \sin \theta' d\theta' = \frac{du}{2r'z} \]

and Eq. (7) becomes

\[ V(z) = \frac{\rho}{2\epsilon_o} \int \int \frac{1}{2zr'} \frac{du}{\sqrt{u}} (r')^2 dr' \]

Now integrate the inner integral over \( u \) and use Eq. (8) to find the appropriate limits of integration as the polar angle goes from \( \theta' = 0 \) to \( \theta' = \pi \). You should obtain the result

\[ V(z) = \left[ \frac{\rho}{2\epsilon_o z} \int \sqrt{(z^2 + r')^2 - (z^2 - r')^2} \right] r' dr' \]

To evaluate this integral only take the positive square roots and be sure that you are inside the sphere. Now you need to be very careful here as each of the two integrals in Eq. (11) splits into two separate regimes: one where \( 0 < r' < z \) and the other where \( z < r' < a \), where now \( z \) is a point that must be inside the sphere. Use this result to get a familiar result for the electric field \( \vec{E} \)

\[ \vec{E} = \int \frac{qr}{4\pi\epsilon_o a^3} \hat{r} \]

where \( q \) is the total charge of the sphere. Note that your answer is identical to the case where the total charge \( q \) is concentrated at the center of the sphere which is neat!

**Problem 3**

Show that the electrostatic potential \( V \) of a uniformly charged disk of surface charge density \( \sigma \) and radius \( a \) at a distance \( z \) on the axis of symmetry is given by

\[ V(0,0,z) = \frac{\sigma}{2\epsilon_o} \left[ \sqrt{(z^2 - a^2)} - z \right] \quad z > 0 \]

\[ V(0,0,z) = \frac{\sigma}{2\epsilon_o} \left[ \sqrt{(z^2 + a^2)} - z \right] \quad z < 0 \]

Carefully discuss its limiting behaviors.

**Problem 4**

Calculate the electric field \( \vec{E} \) everywhere at a distance \( z \) from the center of a spherical surface of radius \( a \) and uniform surface charge density \( \sigma \). Do not use the electrostatic potential at but use Problem 1 of Problem Set VI as a template for solving this problem.
Problem 5

Find the electrostatic potential $V$ of a uniformly charged solid cylinder at a distance $z$ from the center. The length of the cylinder is $L$, its radius is $a$, and its uniform volume charge density is $\rho_v$.

$$V(z) = \frac{\rho_v}{4\varepsilon_0} \left[ -2zL + \left( z + \frac{L}{2} \right) \sqrt{\left( z + \frac{L}{2} \right)^2 + a^2} - \left( z - \frac{L}{2} \right) \sqrt{\left( z - \frac{L}{2} \right)^2 + a^2} \right] + \frac{\rho_v}{4\varepsilon_0} a^2 \ln \frac{(z + \frac{L}{2}) + \sqrt{\left( z + \frac{L}{2} \right)^2 + a^2}}{(z - \frac{L}{2}) + \sqrt{\left( z - \frac{L}{2} \right)^2 + a^2}}$$

Hints:

1. Carefully design your figure and its parameters $z$, $z'$, and $\vec{r} - \vec{r}'$ in this problem and use your figure to relate them. Think about how cylindrical polar coordinates work! Remember that $z$ is always fixed in this problem!

2. You will encounter the following integral to evaluate. First, express it as follows and use a suitable choice of integration by parts to expand it out further

$$\int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

3. You will need to use the following relationship to simplify your integral on the right-side

$$1 + \tan^2 x = \sec^2 x$$

4. Given this result you can find a new expression for

$$\int \sec^3 x \, dx$$

5. In this new expression you will encounter an old friend to evaluate

$$\int \sec x \, dx$$

6. You should be able to handle things from here!
Problem 1

Starting with (2)

\[ u = z^2 + (r')^2 - 2zr' \cos \theta' \]

we realize that \( z \) is fixed and \( r' \) is also fixed in this expression when we integrate over \( \theta \) as seen in our original figure.

\[ \text{du} = 2zr' \sin \theta' \text{d} \theta' \]

This

\[ \sin \theta' \text{d} \theta' = \frac{1}{2zr'} \text{du} \]

which is (3) and (1) becomes

\[ V = \frac{P}{2 \pi v} \int \int \frac{1}{2zr'} \text{du} \frac{(r')^2}{\sqrt{u}} \text{d}r' \]

where we have used \( u = z^2 + (r')^2 - 2zr' \cos \theta' \).
Now integrate over $u$

$$V(z) = \frac{1}{4\epsilon_0 2\pi} \int\! \int\! (r')^2 \, dr' \, u^{-\frac{1}{2}} \, du$$

or

$$V(z) = \frac{1}{4\epsilon_0 2\pi} \int\! \int\! u^{-\frac{1}{2}} \, du \, r' \, dr'$$

Using $u = z^2 + (r')^2 - 2yr' \cos \theta'$

to obtain the limits of integration

\[
\begin{array}{c|c}
\theta' & u \\
\hline
0 & (z-r')^2 \\
\pi & (z+r')^2 \\
\end{array}
\]

$$\int u^{-\frac{1}{2}} \, du = \int u^{-\frac{1}{2}+1} = 2 \sqrt{u}$$

$$\int u^{-\frac{1}{2}} \, du = 2 \left[ \sqrt{(z+r')^2} - \sqrt{(z-r')^2} \right]$$

and $V(z)$ becomes
\[ V(z) = \frac{\rho}{4\pi} \int \frac{1}{r} \left[ \sqrt{(z+r')^2 - (z-r')^2} \right] r' \, dr' \]

or

\[ V(z) = \frac{\rho}{2\pi} \int \sqrt{(z+r')^2 - (z-r')^2} \, r' \, dr' \]

We want \( z > r' \) in this problem so we must take the positive square roots:

\[ \sqrt{(z+r')^2} = z + r' \]

\[ \sqrt{(z-r')^2} = z - r' \quad (z > r') \]

and we find

\[ V(z) = \frac{\rho}{2\pi} \int \left[ (z+r') - (z-r') \right] r' \, dr' \]

\[ V(z) = \frac{\rho}{2\pi} \int 2r' \, r' \, dr' = \frac{\rho}{6\pi} \left( \frac{r'^3}{3} \right)_0^4 \]
\[ V(z) = \frac{4}{6\pi} \rho \frac{a^3}{z^3} \]

Since \( \rho = \frac{3g}{4\pi a^3} \)

\[ V(z) = \frac{3g a^2}{326.4\pi a^3} = \frac{g}{416.4z} \]

or

\[ V(r) = \frac{1}{416.4r}, \quad r > a \]

which makes sense!

Of course,

\[ \vec{E} = -\nabla V(r) = \frac{g}{416.4r^2} \hat{r} \]

which also checks out!
Problem 2

Starting with (7)

\[ V = z^2 + (r')^2 - 2z r' \cos \theta' \]

we realize that \( z \) is fixed and \( r' \) is also fixed in this expression when we integrate over \( \theta \) as seen in our original figure.

\[ dv = 2z r' \sin \theta' \, d\theta' \]

Thus

\[ \sin \theta' \, d\theta' = \frac{1}{2z r'} \, dv \]

so this is (8) so (7) becomes

\[ V = \frac{1}{2z} \int \int \frac{1}{2z r'} \frac{dv}{dv} \ (r')^2 \, dr' \]

when we have used

\[ v = z^2 + (r')^2 - 2z r' \cos \theta' \]
From this definition for $v$ we can do the inner integral over $u$ and evaluate its limits of integration:

\[ \begin{array}{c|c}
\theta^1 & v \\
\hline
0 & (z - r')^2 \\
\pi & (z + r')^2 \\
\end{array} \]

\[
\int u^{-\frac{1}{2}} \, du = \int u^{-\frac{1}{2}} \, du = u^{-\frac{1}{2}} + \frac{1}{2} \left[ \frac{(z + r')^2}{(z - r')^2} \right]
\]

\[= 2 \left[ \sqrt{(z + r')^2} - \sqrt{(z - r')^2} \right] \]

and $V(z)$ becomes

\[
V(z) = \frac{P}{(2\pi e)} \int \left[ \sqrt{(z + r')^2} - \sqrt{(z - r')^2} \right] r' \, dr'
\]

Now there are two regions over which we do this integral.
Inside the sphere $0 < r' < z$

and $z < r' < a$

For $0 < r' < z$, we must have $z > r'$

so

$$\sqrt{(z + r')^2} = r(z + r')$$

$$\sqrt{(z - r')^2} = + (z - r'), \text{ so } z > r'$$

$$V(z) = \frac{\rho}{2\pi \varepsilon_0} \int_0^z [r(z + r') - (z - r')] r' dr'$$

$$V(z) = \frac{\rho}{2\pi \varepsilon_0} \int_0^z 2(r')^2 dr' = \frac{\rho}{2\pi \varepsilon_0} \frac{z^3}{3}$$

For $z < r' < a$ we must have $a > r' > z$

so

$$\sqrt{(z + r')^2} = + (z + r')$$

$$\sqrt{(z - r')^2} = \sqrt{(r' - z)^2} = + (r' - z), \text{ for } r' > z$$
Our integral now becomes

\[ V(z) = \frac{\rho}{2\pi \epsilon_0} \int \frac{(z + r') - (z' - z)}{z} r' \, dr' \]

\[ V(z) = \frac{\rho}{2\pi \epsilon_0} \int \frac{3z + r'}{z} r' \, dr' \]

\[ V(z) = \frac{\rho}{6\epsilon_0} \left. \frac{(r')^2}{2} \right|_0^a \]

\[ V(z) = \frac{\rho}{2\pi \epsilon_0} \sqrt{a^2 - z^2} \]

Combining our two results for \( 0 < z < a \) and \( z < r' < a \), we find

\[ V(z) = \frac{\rho z^2}{36\epsilon_0} - \frac{\rho a^2}{2\pi \epsilon_0} - \frac{\rho z^2}{2\pi \epsilon_0} \]

\[ V(z) = \frac{2\rho z^2}{6\epsilon_0} - \frac{3\rho z^2}{6\epsilon_0} + \frac{\rho a^2}{2\pi \epsilon_0} \]

\[ V(z) = \frac{\rho a^2}{2\pi \epsilon_0} - \frac{\rho z^2}{6\epsilon_0} \]
\[ V(z) = \frac{\rho}{2\varepsilon_0} \left[ \frac{a^2 - z^2}{3} \right], \quad z < a \]

Note where \( z = a \)

\[ V(a) = \frac{\rho}{2\varepsilon_0} \left[ \frac{a^2 - a^2}{3} \right] = \frac{\rho a^2}{3 \varepsilon_0} \]

and for \( \rho = \frac{g}{4\pi a^3} \)

\[ V(a) = \frac{3\frac{g}{4\pi a^3}}{3 \varepsilon_0} \frac{a^2}{3 \varepsilon_0} = \frac{g}{4\pi \varepsilon_0 a} 
\]

which makes sense!
Finally, if

$$V(z) = \rho \left[ a^2 - \frac{z^2}{3} \right]$$

we can express our result entirely in terms of $q$

$$\rho = \frac{q}{4\pi a^3}$$

$$V(z) = \frac{q^3}{4\pi a^3} \frac{1}{2a} \left[ a^2 - \frac{z^2}{3} \right]$$

$$V(z) = \frac{3q}{8\pi a^3} \frac{1}{2a} \left[ a^2 - \frac{z^2}{3} \right]$$

$$V(z) = \frac{9}{8\pi \varepsilon_0 a} \left[ 3 - \frac{z^2}{a^2} \right]$$

For $z < a$

What is $\vec{E}$?

$$\frac{\partial V}{\partial z} = -\frac{2z}{a^2} \frac{q}{8\pi \varepsilon_0 a} = -\frac{9q}{4\pi \varepsilon_0 a^3}$$

$$\vec{E} = \frac{9q}{4\pi \varepsilon_0 a^3} \hat{z}$$

which makes perfect sense!
Problem 3

Starting with our coordinate system in cylindrical polar coordinates,

we can use them for our problem

\[
V(0, 0, z) = \frac{1}{4\pi \epsilon_0} \int \int \int \frac{\rho \, d\rho \, d\phi \, dz}{(\rho^2 + z^2)^{1/2}}
\]

and proceed

\[
V(0, 0, z) = \frac{\sigma}{4\pi \epsilon_0} \int \rho \, d\rho \, d\phi \, dz \quad (4-2)
\]
From our figure

\[ |\vec{r} - \vec{r}'| = \left[ \rho^2 + z'^2 \right]^{\frac{1}{2}} \quad (4.3) \]

so \((4.2)\) becomes

\[ V(z) = \frac{\sigma}{4\pi \varepsilon_0} \int \frac{\rho \, d\rho \, d\phi}{\left[ \rho^2 + z^2 \right]^{\frac{3}{2}}} \quad (4.4) \]

let us integrate out the azimuthal angle

\[ V(z) = \frac{\sigma}{2\pi \varepsilon_0} \int \frac{\rho \, d\rho}{\left[ \rho^2 + z^2 \right]^{\frac{3}{2}}} \quad (4.5) \]

Now since \(z\) is fixed we are in business,

\[ u = \rho^2 + z^2 \quad (4.6) \]

\[ du = 2\rho \, d\rho \quad (4.7) \]

and

\[
\begin{array}{c|c}
\rho & u \\
\hline
0 & z^2 \\
\alpha & z^2 + \alpha^2 \\
\end{array}
\]
So we can take into account our limits of integration

\[(4.8) \quad V(z) = \frac{\sigma}{2\pi\epsilon_0} \int \frac{1}{2\sqrt{u}} \left[ \frac{u^{-\frac{1}{2}}}{\sqrt{u}} \right] \frac{z^2}{2} \frac{z^2}{u^2} du \]

\[(4.9) \quad V(z) = \frac{\sigma}{2\pi\epsilon_0} \left[ \left( \frac{z^2}{r^2} \right)^{\frac{3}{2}} - \left( \frac{z^2}{r^2} \right)^{\frac{1}{2}} \right] \]

Note that in our figure \(z > 0\), so

\[(4.10) \quad V(z) = \frac{\sigma}{2\pi\epsilon_0} \left[ \left( \frac{z^2}{r^2} + a^2 \right)^{\frac{3}{2}} - z \right], \quad z > 0 \]

By symmetry, the electrostatic potential \(V(z)\) must have the same value for \(z < 0\).

If we take the negative square root of \((z^2)^{\frac{3}{2}}\) (as opposed to the positive square root) as we did in \((4.9)\), we have

\[(4.11) \quad (z^2)^{\frac{3}{2}} = -z \]

which must give the same physical result for

\[(4.12) \quad z < 0 \]
\[ V(z) = \frac{\sigma}{2\varepsilon_0} \left[ (z^2 + a^2)^{\frac{\mu}{2}} + z \right], \quad z < 0 \]

Thus

\[ V(z) = \frac{\sigma}{2\varepsilon_0} \left[ (z^2 + a^2)^{\frac{\mu}{2}} - z \right], \quad z > 0 \]

\[ V(z) = \frac{\sigma}{2\varepsilon_0} \left[ (z^2 - a^2)^{\frac{\mu}{2}} + z \right], \quad z < 0 \]

Finally, let us explore the limiting cases.

1. Note for \( z = 0 \):
\[ V = \frac{\sigma a}{2\varepsilon_0} = \frac{q}{\pi a^2} \frac{a}{2\varepsilon_0} = \frac{q}{2\pi\varepsilon_0 a} \]

2. For \( z \gg a \):
\[ \left[ (z^2 + a^2)^{\frac{\mu}{2}} - z \right] = \left[ z \left( 1 + \left( \frac{a}{z} \right)^2 \right)^{\frac{\mu}{2}} - z \right] \]
\[ = z \left[ \left( 1 + \left( \frac{a}{z} \right)^2 \right)^{\frac{\mu}{2}} - 1 \right] \]
You can expand
\[(1 + x)^{\frac{1}{2}}\]
as a binomial series
\[(1 + \frac{1}{2}x + \frac{1}{2!} \frac{1}{2}(\frac{1}{2} - 1)x^2 + \cdots)\]
which for small values of \(x\) only becomes important for lower order terms
\[(1 + x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x + \cdots\]

Thus for \(a < Z\)
\[\left[ (Z^2 + a^2)^{\frac{1}{2}} - z \right] = \left[ (1 + \frac{1}{2}(\frac{a}{Z})^2)^{\frac{1}{2}} - 1 \right] \]
\[\approx \left[ 1 + \frac{1}{2} \left( \frac{a}{Z} \right)^2 \cdots - 1 \right] \]
\[= \frac{1}{2} \left( \frac{a}{Z} \right)^2\]
and
\[\left[ (Z^2 + a^2)^{\frac{1}{2}} - z \right] \approx \frac{Z}{2} \left( \frac{1}{2} \frac{a^2}{Z^2} \right) \approx \frac{a^2}{2Z}\]
\[ V(z) = \frac{\sigma}{2\epsilon_0} \frac{a^2}{2z} = \frac{\sigma}{4\epsilon_0} \frac{a^2}{z} \]

or

\[ V(z) = \frac{\sigma}{4\epsilon_0} \frac{a^2}{z} = \frac{\sigma a^2}{4\epsilon_0 \epsilon_r} \frac{z}{2} \]

\[ V(z) \rightarrow \frac{g}{4\epsilon_0 \epsilon_r z} \quad \text{as} \quad z \rightarrow a \]

which makes perfect sense.

We shall note that the binomial series is just an example of a Taylor series. If you need a review of

1. Maclaurin series
2. Taylor series
3. Binomial expansion
4. Binomial series

I would strongly recommend Khan Academy for an excellent discussion of these topics!
Problem 4

Find $E$ everywhere along $z$!

Using Problem 1 of Problem Set VI as a template, symmetry says there is only one component that contributes to $E$

\begin{equation}
E_z = \int \frac{d\theta \cos \theta}{4\pi \mu \sqrt{z^2 - r^2}}
\end{equation}

where

\begin{equation}
\sqrt{z^2 - r^2} = s \quad s^2 = z^2 + a^2 - 2az \cos \theta
\end{equation}

and

\begin{equation}
a^2 = z^2 + s^2 - 2zs \cos \alpha
\end{equation}

(N.B. $s \neq S$ $L \arccos \frac{1}{17}$)
It follows from spherical polar coordinates that

\[(4-4) \quad d\mathbf{q} = \sigma a^2 \sin \theta \, d\theta \, d\phi = \sigma \, d\mathbf{s}\]

and

\[(4-5) \quad E_z = \int \sigma a^2 \, \frac{\sin \theta \, d\theta \, d\phi \, \cos \lambda}{\sqrt{\lambda}} \, \frac{d\mathbf{s}}{s^3}\]

To simplify the integrand a little we observe from \((4-3)\) that

\[(4-6) \quad \cos \lambda = \frac{z^2 + s^2 - a^2}{2zs}\]

and upon doing the azimuthal integral in \((4-5)\) we find

\[(4-7) \quad E_z = \frac{\sigma a^2}{4\pi \lambda^2} \int \frac{\sin \theta \, d\theta \, \left[\frac{z^2 + s^2 - a^2}{2zs}\right] \, d\mathbf{s}}{s^3}\]

where \(z\) is fixed and we only have two variables of integration \(\theta, s\)
If we perform the polar integration over $\theta$, then we obtain from (4-2)

\[
(4-8) \quad -2\pi z \cos \theta = s^2 - z^2 - a^2
\]

or

\[
(4-9) \quad \sin \theta d\theta = ds \left[ \frac{s^2 - z^2 - a^2}{2\pi z} \right] = \frac{2s ds}{2\pi z}
\]

or

\[
(4-10) \quad \sin \theta d\theta = \frac{5 ds}{a z}
\]

so (4-7) simplifies to

\[
(4-11) \quad E_z = \sigma \frac{a^2}{4 \epsilon_0 z^2} \int ds \left[ \frac{s^2 - z^2 - a^2}{a z s^2} \right]
\]

or

\[
(4-12) \quad E_z = \sigma a \frac{z^2 - a^2 + 1}{4 \epsilon_0 z^2} \int ds \left[ \frac{s^2}{z^2} + 1 \right]
\]

Now we need to find the limits of integration over $s$. 

Now we need to find the limits of integration over $s$.
From (4.12) we have

\[ (4.15) \quad S^2 = z^2 + a^2 - 2az \cos \Theta \]

and from our figure we see that we are integrating \( \Theta \) from 0 to \( \pi \) so that

For \( \Theta = 0 \) \[ S^2 = (z-a)^2 \] (4.14)

For \( \Theta = \pi \) \[ S^2 = (z+a)^2 \] (4.15)

We first will evaluate (4.12) for \( z > a \) so we take the positive square roots to find

\[ (4.16) \quad S = (z-a) \quad \text{for} \quad \Theta = 0 \]

\[ (4.17) \quad S = (z+a) \quad \text{for} \quad \Theta = \pi \]

and (4.12) becomes

\[ (4.18) \quad E_2 = \frac{\delta a}{4G\pi z^2} \int_{\frac{z^2-a^2}{z^2}}^{\frac{z^2+a^2}{z^2}} \left[ \frac{z^2-a^2}{S^2} + 1 \right] \, ds \]
This integral is now straightforward to do.

\[(4.19) \quad \mathbf{E}_2 = \sigma a \left[ \frac{-(z^2-a^2)}{z-a} \right] \]

or

\[(4.20) \quad \mathbf{E}_2 = \sigma a \left[ \frac{(z+a) - (z^2-a^2)}{z+a} \right] - \left( \frac{z-a}{z-a} \right) - \left( \frac{z^2-a^2}{z-a} \right) \]

or

\[(4.21) \quad \mathbf{E}_2 = \sigma a \left[ \frac{z+a - (z-a)}{z+a} \right] - (z-a) + z+a \]

or

\[(4.22) \quad \mathbf{E}_2 = \sigma a \left[ \frac{z-a+z+a}{z+a} \right] \]

or

\[(4.23) \quad \mathbf{E}_2 = \frac{\sigma a^2}{60z^2} \]

or

\[(4.23) \quad \mathbf{E}_2 = \frac{\sigma a^2}{60z^2} \]
Finally since
\[ \delta = \frac{g}{4\pi a^2} \]

\[(4.24) \quad E_z = \frac{g}{4\pi \varepsilon_0 Z^2} \]

or
\[ \sqrt{(4.28) \quad E = \frac{1}{4\pi \varepsilon_0 r^2}, \text{ for } r > a} / \]

which makes sense!

Now we return to (4.14) to study the case where \( z < a \). In order to have a positive square root we must realize

\[(4.29) \quad \text{For } \theta = 0 \quad s^2 = (z-a)^2 \]

\[ s = \sqrt{(z-a)^2} = \sqrt{(a-z)^2} = a-z \]

\[(4.30) \quad \text{For } \theta = \pi \quad s^2 = (z+a)^2 \]

\[ s = \sqrt{(z+a)^2} = z+a \]
and (4-19) becomes

\[
(4-31) \quad E_2 = \frac{\sigma a}{4 \varepsilon_0 z^2} \left[ s - \frac{(z^2 - a^2)}{s} \right] \frac{(z+a)}{(a-z)}
\]

Evaluating this yields

\[
(4-32) \quad E_2 = \frac{\sigma a}{4 \varepsilon_0 z^2} \left[ \frac{(z+a) - (z^2 - a^2)}{(z+a)} \right. \\
- (a-z) \left. + \frac{(z^2 - a^2)}{(a-z)} \right]
\]

or

\[
(4-33) \quad E_2 = \frac{\sigma a}{4 \varepsilon_0 z^2} \left[ (z+a) - (z-a) \right.
\]

\[- (a-z) - (a+z) \left. \right]
\]

\[
(4-34) \quad E_2 = \frac{\sigma a}{4 \varepsilon_0 z^2} \left[ \frac{1}{z-z} \right.
\]

\[\left. + \frac{1}{a+z+a} - \frac{1}{a+z} + \frac{1}{z} \right]
\]

\[
(4-35) \quad \bar{E} = 0, \quad z < a \quad (\text{which makes sense from Gauss's law})
\]
Problem 5

Find the electrostatic potential $V$ on the axis of a uniformly charged solid cylinder, a distance $z$ from the center. The length of this cylinder is $L$, its radius $a$, and its charge density $\rho_V$. Use your assumption that $z > \frac{1}{2}$ in this problem carefully!

Let us exploit cylindrical polar coordinates to study the problem and draw the solid cylinder carefully.

Some observations:

1. $\rho_V = \text{volume charge density}$
2. $\rho = \text{coordinate in cylindrical polar coordinates}$
3. $\rho \neq \rho_V$
4) The cylinder is centered at \((0, 0, 0)\).

5) The distance \(z\) from the origin to a point outside the cylinder is a constant.

6) The cylinder has a height from \(z = -\frac{L}{2}\) to \(z = +\frac{L}{2}\) or \(L\).

7) The radius of the cylinder is a (constant).

Now let us be careful!

8) The differential volume element of \(dV\) is

\[ dV' = \rho' \, dp' \, dq' \, dz' \]

where we use primed variables to refer to the charge distribution. Note \(z'\) is not constant and it tells you where \(dV'\) is located with respect to the origin!

9) How are \(\rho', z',\) and \(z\) related?
Look carefully at our figure!

Note that \( z \) is greater than \( \frac{1}{2} \) or outside the cylinder. We will return to this point later.

From our above figure:

\[
|z-z'| = \left[ (\rho')^2 + (z-z')^2 \right]^{1/2}
\]

This makes sense as for \( z'=z \)

\[
|z-z'| = \rho'
\]

and for \( z'=0 \)

\[
|z-z'| = \sqrt{(\rho')^2 + z^2}^{1/2}
\]

Clearly \( z' + (z-z') = z \)

Make sure you see this from figure!!
19 Finally we can modify (3). The variable \( z' \) must give us information about the height of the cylinder so it goes from \( x' = -\frac{a}{2} \) to \( x' = \frac{a}{2} \).

Now we are ready to solve our problem.

\[ V = \int \frac{dz}{4\pi \varepsilon_0} = \int_0^a \frac{\rho' dp' d\rho' dz'}{4\pi \varepsilon_0 \sqrt{[(\rho')^2 + (z-z')^2]^{\frac{3}{2}}}} \]

Doing the azimuthal angle first gives

\[ V(z) = \frac{2\pi}{2\pi} \int_0^a \frac{\rho' dp' dz'}{\sqrt{[(\rho')^2 + (z-z')^2]^{\frac{3}{2}}}} \]

or

\[ V(z) = \frac{\rho'}{2\pi} \int_0^a \frac{dp' dz'}{\sqrt{[(\rho')^2 + (z-z')^2]^{\frac{3}{2}}}} \]

Let us evaluate the inner integral over \( \rho' \) first.
$$V(z) \sim \int \frac{\rho \, dp'}{\sqrt{L(p')^2 + (z-z')^2}}$$

where \( z \) is of course a constant and we hold \( z' \) fixed while performing this integral.

Let us define

$$u = (p')^2 + (z-z')^2$$

where

$$du = 2(p') dp'$$

Our integral (inner) becomes

$$\frac{1}{2} \int \frac{du}{u^{1/2}} = \frac{1}{2} \left[ u^{-1/2 + 1} \right] = \frac{\sqrt{u}}{1 - \frac{1}{2}}$$

Now we need to evaluate the limits of integration for \( u \).
Su we can evaluate (7) and place it back into (3) to yield

\[ V(z) = \frac{\varphi V}{2 \kappa_0} \int \left[ \frac{1}{\sqrt{(z-z')^2 + \kappa^2}} - \sqrt{(z-z')^2} \right] \, dz' \]

or

\[ V(z') = \frac{\varphi V}{2 \kappa_0} \int \left[ \frac{1}{\sqrt{(z-z')^2 + \kappa^2}} - \sqrt{(z-z')^2} \right] \, dz' \]

For the moment let us ignore the limits of integration and tackle the first integral

\[ \int \sqrt{(z-z')^2 + \kappa^2} \, dz' \]

Let \( u = z - z' \) \( (11) \) \( du = -dz' \) \( (12) \)

and we get

\[ -\int \sqrt{u^2 + \kappa^2} \, du \]
Now we need to evaluate this integral

\[14\] \quad - \int \sqrt{u^2 + a^2} \, du

where \(a\) is a constant. Let us try trigonometric substitution

\[- \int a \sqrt{1 + \frac{u^2}{a^2}} \, du\]

\[\tan \theta = \frac{u}{a} \quad u = a \tan \theta \quad du = a \sec^2 \theta \, d\theta\]

\[\left[1 + \tan^2 \theta\right]^{1/2} = \sec \theta\]

\[-\int a \sqrt{1 + \frac{u^2}{a^2}} \, du = -\int a \sec \theta \, a \sec^2 \theta \, d\theta\]

\[= -a^2 \int \sec^3 \theta \, d\theta\]

This integral is a bit challenging but we can use a few clever observations
\[
\int \sec^2 \theta \, d\theta = \int \sec \theta \sec^2 \theta \, d\theta
\]

Now integrate by parts.

Let \( u = \sec \theta \)

\( \, du = \sec^2 \theta \, d\theta \)

\( v = \tan \theta \)

\( \, dv = \sec^2 \theta \, d\theta \)

So

\[
\int \sec^2 \theta \, d\theta = \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta \, d\theta.
\]

Now using the fact that

\[ 1 + \tan^2 \theta = \sec^2 \theta \]

our last integral becomes

\[
\int \sec^2 \theta \, d\theta = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta
\]

\[ + \int \sec \theta \, d\theta \]

or
\[ 2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln \left[ \tan \theta + \sec \theta \right] \]

Finally we have seen this last integral before in this course.

\[ 2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln \left[ \tan \theta + \sec \theta \right] \]

So

\[ \int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \sec \theta \tan \theta + \ln \left[ \tan \theta + \sec \theta \right] \right] \]

We can get rid of the trigonometric functions as follows.

\[ \tan \theta = \frac{V}{a} \]

\[ \sin \theta = \frac{V}{(v^2 + a^2)^{1/2}} \]

\[ \cos \theta = \frac{a}{(v^2 + a^2)^{1/2}} \]

So that we have
\[
\int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \frac{u}{a} \right]^{(u^2 + a^2)^{\frac{1}{2}}} + \ln \left[ \frac{u}{a} + \frac{(u^2 + a^2)^{\frac{1}{2}}}{a} \right]
\]

and using (11)

(15) \[
\int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \left( \frac{z-z'}{a} \right)^2 \frac{[(z-z')^2 + a^2]^\frac{1}{2}}{a} \right. \\
+ \left. \ln \left[ \frac{(z-z')}{a} + \frac{[(z-z')^2 + a^2]^\frac{1}{2}}{a} \right] \right]
\]

Thus our first integral (11) becomes

(16) \[
\int \sqrt{(z-z')^2 + a^2} \, dz' \quad = \quad -a^2 \int \sec^3 \theta \, d\theta
\]

\[
= -\frac{a^2}{2} \left[ \left( \frac{z-z'}{a} \right)^2 \frac{[(z-z')^2 + a^2]^\frac{1}{2}}{a} \right. \\
+ \left. \ln \left[ \frac{(z-z')}{a} + \frac{[(z-z')^2 + a^2]^\frac{1}{2}}{a} \right] \right]
\]
Now we are in a position to evaluate the limits of integration of \( z' \) from \(-\frac{L}{2}\) to \(+\frac{L}{2}\).

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{z'^2 + a^2} \, dz' = \frac{a^2}{2} \left[ \left( \frac{z - \frac{L}{2}}{a} \right) \left( \frac{\left( z - \frac{L}{2} \right)^2 + a^2}{a} \right) \right]^{\frac{L}{2}} + \ln \left[ \frac{z - \frac{L}{2}}{a} + \frac{\left( z - \frac{L}{2} \right)^2 + a^2}{a} \right]^{\frac{L}{2}} + \frac{a^2}{2} \left[ \left( \frac{z + \frac{L}{2}}{a} \right) \left( \frac{\left( z + \frac{L}{2} \right)^2 + a^2}{a} \right) \right]^{\frac{L}{2}} + \ln \left[ \frac{z + \frac{L}{2}}{a} + \frac{\left( z + \frac{L}{2} \right)^2 + a^2}{a} \right]^{\frac{L}{2}}
\]
or

\[ \frac{a^2}{2} \left[ \left( \frac{z + \frac{1}{2}}{a} \right) \left( \frac{\left[ \left( z + \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{a} \right) \right. \]

\[ - \frac{(z - \frac{1}{2})}{a} \left( \frac{\left[ \left( z - \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{a} \right) \]

\[ + \ln \left[ \frac{(z + \frac{1}{2}) + \left[ \left( z + \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{(z - \frac{1}{2}) + \left[ \left( z - \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}} \right] \]

Thus the contribution of the first integral by (1) is

\[ \frac{\rho \sigma a^2}{6 \gamma} \left[ \left( \frac{z + \frac{1}{2}}{a} \right) \left( \frac{\left[ \left( z + \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{a} \right) \right. \]

\[ - \frac{(z - \frac{1}{2})}{a} \left( \frac{\left[ \left( z - \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{a} \right) \]

\[ + \ln \left[ \frac{(z + \frac{1}{2}) + \left[ \left( z + \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}}{(z - \frac{1}{2}) + \left[ \left( z - \frac{1}{2} \right)^2 + \frac{a^2}{a} \right]^{\frac{1}{2}}} \right] \]
The second integral is ③ can easily be evaluated as follows

\[ \frac{- PV}{2 \varepsilon_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{(z - z')^2} \, dz' \]

\[ U = z - z' \Rightarrow dz' = -du \] since \( z \) is a constant

\[ \sqrt{(z - z')} = \sqrt{u^2} = u \]

but we take the positive square root so \( z > z' \)

\[ \frac{1}{2 \varepsilon_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{(z - z')^2} \, dz' \]

\[ \frac{1}{2 \varepsilon_0} \int \sqrt{u^2} \, du \]

\[ \begin{array}{c|c|c}
 z' & U & \\
 \hline
 -\frac{1}{2} & z + \frac{1}{2} \\
 \frac{1}{2} & z - \frac{1}{2} \\
\end{array} \]
\[ 2 \left( \frac{p_r}{\sigma_0} \right) \frac{v^2}{2} \frac{1}{z + \frac{1}{2}} \]

\[ \frac{p_r}{2\sigma_0(\lambda)} \left[ (z - \frac{1}{2})^2 - (z + \frac{1}{2})^2 \right] \]

\[ \frac{p_r}{2\sigma_0} \left[ \frac{z^2 + \frac{l^2}{4} - zL - \left( \frac{z^2 + \frac{l^2}{4} + zL}{} \right) \right] \]

\[ \frac{p_r}{2\sigma_0} \left[ -2zL \right] \]

If we combine 25 + 19 we obtain the final result.
\[ V(z) = \frac{\rho \sigma}{4\nu} \left[ (z + \frac{4\nu}{a}) \left[ (z + \frac{4\nu}{a})^2 + a^2 \right]^{\frac{3}{2}} - (z - \frac{4\nu}{a}) \left[ (z - \frac{4\nu}{a})^2 + a^2 \right]^{\frac{3}{2}} \right] \]

\[ + a^2 \ln \left[ \frac{(z + \frac{4\nu}{a}) + \sqrt{(z + \frac{4\nu}{a})^2 + a^2}}{(z - \frac{4\nu}{a}) + \sqrt{(z - \frac{4\nu}{a})^2 + a^2}} \right] \]