USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET I
(due Friday, June 26, 2020)

Problem 1
Show that the two vectors $\vec{u} = (3, 4)$ and $\vec{v} = (4, -3)$ are orthogonal. Draw these two vectors in a cartesian coordinate system.

Problem 2
Prove that the line from the apex of an isosceles triangle that bisects its base is perpendicular to the base.

\[ \text{Diagram of isosceles triangle with line m}} \]
Problem 3
Given the two vectors $\vec{u} = \hat{i} + 2\hat{j}$ and $\vec{w} = 2\hat{i} + \hat{j}$, determine the angle $\phi$ in the figure below.

Problem 4
Find the angles between the two vectors $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 2\hat{i} - 3\hat{j} - \hat{k}$.

Problem 5
Show that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$.

Problem 6
Evaluate $\vec{u} \times \vec{v}$ if $\vec{u} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{v} = 2\hat{i} + 2\hat{j} - \hat{k}$.

Problem 7
Prove the following property of a vector product

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Problem 8
Prove the following property of a vector product

$$(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = -c\vec{u} \times \vec{v}$$

Problem 9
Prove the following property of a vector product

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
Problem 10
Prove the triple scalar product shown below is true

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

Problem 11
Prove that the triple vector product of the following three vectors satisfies the famous rule below

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Problem 12
Show that the three vectors $\vec{u} = \hat{i} + \hat{j} - \hat{k}$, $\vec{v} = 2\hat{j} + \hat{k}$, and $\vec{w} = 2\hat{i} + 4\hat{j} - \hat{k}$ are coplanar.

Problem 13
Prove that in plane polar coordinates

$$\hat{i} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi}$$

$$\hat{j} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi}$$

Problem 14
Show that the unit vectors for spherical polar coordinates are given by

$$\hat{i} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\hat{j} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$$

$$\hat{k} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$
This coordinate system is called a *spherical coordinate system* because the graph of the equation $r = c = \text{constant}$ is a sphere of radius $c$ centered at the origin.

Occasionally we need to know $r, \theta$, and $\phi$ in terms of $x, y,$ and $z$. These relations are given by (Problem D-1)

$$r = \left( x^2 + y^2 + z^2 \right)^{1/2}$$

$$\cos \theta = \frac{z}{\left( x^2 + y^2 + z^2 \right)^{1/2}}$$

$$\tan \phi = \frac{y}{x} \quad (D.2)$$

Any point on the surface of a sphere of unit radius can be specified by the values of $\theta$ and $\phi$. The angle $\theta$ represents the declination from the north pole, and hence $0 \leq \theta \leq \pi$. The angle $\phi$ represents the angle about the equator, and so $0 \leq \phi \leq 2\pi$. Although there is a natural zero value for $\theta$ (along the north pole), there is none for $\phi$. Conventionally, the angle $\phi$ is measured from the $x$-axis as illustrated in Figure D.1. Note that $r$, being the distance from the origin, is intrinsically a positive quantity. In mathematical terms, $0 \leq r < \infty$.

In Chapter 6, we will encounter integrals involving spherical coordinates. The differential volume element in Cartesian coordinates is $dxdydz$, but it is not quite so