

## All Majorana Models with Translation Symmetry are Supersymmetric

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We establish results similar to Kramers and Lieb-Schultz-Mattis theorems but involving only translation symmetry and for Majorana modes. In particular, we show that all states are at least doubly degenerate in any one- and two-dimensional array of Majorana modes with translation symmetry, periodic boundary conditions, and an odd number of modes per unit cell. Moreover, we show that all such systems have an underlying  $\mathcal{N} = 2$  supersymmetry and explicitly construct the generator of the supersymmetry. Furthermore, we establish that there cannot be a unique gapped ground state in such one-dimensional systems with antiperiodic boundary conditions. These general results are fundamentally a consequence of the fact that translations for Majorana modes are represented projectively, which in turn stems from the anomalous nature of a single Majorana mode. An experimental signature of the degeneracy arising from supersymmetry is a zero-bias peak in tunneling conductance.

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A Majorana mode is a strange concept. Formally, it represents  $\sqrt{2}$  degree(s) of freedom because two Majorana modes constitute a single qubit or spinless fermion. By construction, the Majorana mode is its own antimode: its creation and annihilation operators are identical [1]. While the mathematical existence of Majorana modes arises simply from a change of basis in the particle-hole space, the physical manifestations of the above properties are extremely nontrivial [2] and important for quantum computation purposes [3–5]. The fact that a Majorana mode is only a fraction of a physical electron or qubit suggests the possibility of encoding information in two widely separated Majorana modes, each of which is immune to local decoherence. Furthermore, the Hermitian nature of the Majorana mode forces it to exist at zero energy in the superconducting gap of physical systems, allowing experimentalists to zero in on finding a zero-bias peak in tunneling, which is necessary if a Majorana mode is present.

There has been tremendous effort [6–10] toward realizing these Majorana modes at the end points of both one-dimensional topological superconductors in nanowires [11–15] and atomic chains [16], as well as from two-dimensional interfaces between topological insulators and superconductors [17–19]. The increasingly compelling evidence for single Majorana modes and the substantial activity in this field suggest that scaling the system to realize multiple Majorana modes along a line or in a two-dimensional grid may be realized in the near future. In such setups involving lattices of emergent Majorana modes, the low energy effective Hamiltonian involves interactions between such modes, and such models host abundant and fascinating phenomenology. For example, two- [20] and three-dimensional [21] lattices of Majorana zero modes may provide new architectures for quantum information processing and new topological phases.

Moreover, there have been several proposals that employ Majorana modes for realizing supersymmetry, which is a highly appealing concept from particle physics [22–25], relating bosonic and fermionic modes to each other. Though signatures of it have yet to be observed, there are several condensed matter systems in which supersymmetry may emerge at long time and distance scales (“scaling limit”), especially close to a critical point [26–30]. In particular, the supersymmetric tricritical Ising model may be realized at a critical point of Majorana systems [28,31]. Furthermore, time reversal acts as a supersymmetry on vortices of topological superconductors [32]. However, exact supersymmetry in lattice models typically requires fine-tuned Hamiltonians [33–38].

In this work, we show that *all* Majorana systems with translation symmetry and an odd number of Majorana modes per unit cell exhibit  $\mathcal{N} = 2$  supersymmetry and we explicitly construct its generator and identify its experimental consequences. For one-dimensional systems with periodic boundary conditions, we establish a Kramers-like theorem [39] but for translation, not time-reversal symmetry: we show that every energy level is at least doubly degenerate. With antiperiodic boundary conditions, we establish a result along the lines of Lieb-Schultz-Mattis [40] and rule out the possibility of a unique gapped ground state, using recent results for spin chains [41–44]. For two-dimensional systems, we also establish at least twofold degeneracy for all states and for all system dimensions. The essence of all these results is the fractional nature of the Majorana mode. Each unit cell, with an odd number of Majorana modes, cannot exist intrinsically, and therefore the symmetry group involving translations and fermion parity is represented projectively. We will now illustrate this in detail and conclude by mentioning several experimental venues

for our results, in which a striking signature of supersymmetry is a zero-bias peak in tunneling experiments.

*1D, periodic boundary conditions.*—Consider  $N$  (even, to have a well-defined total Hilbert space) Majorana modes  $\{\gamma_i\}_{i=1,\dots,N}$  localized at  $N$  sites spaced around a ring (Fig. 1), and consider general Hamiltonians invariant under translation by one site and with periodic boundary conditions:

$$H = \sum_{i=1}^N H_i. \quad (1)$$

We make no assumptions on the structure of  $H_i$  other than translation invariance and the conservation of fermion parity. In particular, let  $\hat{P} \equiv i^{N/2} \prod_{i=1}^N \gamma_i$  be the fermion parity operator, for which we have  $[\hat{P}, H_i] = [\hat{P}, H] = 0$ .

Let  $T$  be translation by one site:

$$\hat{T}\gamma_i\hat{T}^{-1} = \gamma_{i+1(\text{mod } N)}. \quad (2)$$

By assumption,

$$\hat{T}H_i\hat{T}^{-1} = H_{i+1(\text{mod } N)} \quad (3)$$

$$\Rightarrow [\hat{T}, H] = 0. \quad (4)$$

However, translation and fermion parity anticommute:

$$\hat{T}\hat{P}\hat{T}^{-1} = i^{N/2} \left( \prod_{i=2}^N \gamma_i \right) \gamma_1 = -\hat{P}, \quad (5)$$

because  $N$  is even and  $\gamma_1$  is anticommutated through an odd number of Majorana modes to return to the beginning. We note that the fact that  $P$  and  $T$  can anticommute has been used to classify fermionic phases protected by translation symmetry [45]. It follows that every eigenstate of  $H$  is at least doubly degenerate: if  $H|\psi\rangle = E|\psi\rangle$  and  $P|\psi\rangle = p|\psi\rangle$  ( $p = \pm 1$ ), then  $HT|\psi\rangle = TH|\psi\rangle = ET|\psi\rangle$  and  $PT|\psi\rangle = -TP|\psi\rangle = -pT|\psi\rangle$ , and thus  $T|\psi\rangle$  is also an eigenstate with energy  $E$  and orthogonal to  $|\psi\rangle$ . A similar algebraic structure (but not involving translation) was used in Ref. [46] to establish spectrum doubling.

*Degeneracy as a consequence of  $\mathcal{N} = 2$  supersymmetry.*—We now show that all translationally invariant Majorana Hamiltonians in 1D with periodic boundary conditions and an odd number of Majorana modes per unit cell are supersymmetric. The twofold degeneracy of the spectra found in the previous section can then be thought of as a consequence of this underlying supersymmetry.

We first shift all the eigenvalues of the Hamiltonian  $H$  by a constant so that they are all non-negative. Then we define the following fermionic, non-Hermitian operator  $\hat{Q}$ :

$$\hat{Q} = \sqrt{\frac{H}{2}} \hat{T} (\hat{1} + \hat{P}), \quad (6)$$

where  $\hat{1}$  is the identity operator. Clearly,  $\hat{Q}$  commutes with the Hamiltonian  $H$ :  $[H, \hat{Q}] = 0$ . Most importantly, due to the relation  $\{\hat{T}, \hat{P}\} = 0$ , one finds

$$\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0, \quad (7)$$

$$\{\hat{Q}, \hat{Q}^\dagger\} = 2H. \quad (8)$$

Therefore,  $\hat{Q}$  acts as the generator of an  $\mathcal{N} = 2$  supersymmetry ( $\mathcal{N}$  equals 2 because  $\hat{Q}$  is a non-Hermitian operator and can be decomposed as  $\hat{Q} = \hat{Q}_1 + i\hat{Q}_2$ , where  $\hat{Q}_1$  and  $\hat{Q}_2$  are Hermitian) [47]. Thus, all Majorana Hamiltonians in 1D that have an odd number of Majorana modes per unit cell with periodic boundary conditions furnish an  $\mathcal{N} = 2$  supersymmetry.

Supersymmetry naturally explains the results derived in the previous section on the nature of spectra. All energy eigenvalues are doubly degenerate, and the corresponding eigenstates can be chosen as fermion parity eigenstates with opposite parity. Explicitly, given an eigenstate  $|n\rangle_B$  with energy  $E_n$  and fermion parity  $+1$ , its fermionic partner eigenstate, which has the same energy eigenvalue  $E_n$  but opposite parity, is given by

$$|n\rangle_F = \frac{\hat{Q}|n\rangle_B}{\sqrt{2E_n}}, \quad (9)$$

where we have normalized so that  ${}_F\langle n|n\rangle_F = 1$ . The opposite fermion parity of  $|n\rangle_F$  follows because  $\{\hat{P}, \hat{Q}\} = 0$ .

Because of the factor of  $1/\sqrt{2E_n}$  in Eq. (9), in a general supersymmetric theory, the existence of supersymmetric partner eigenstates is guaranteed only when  $E_n \neq 0$ . However, in our case, the zero of energy plays no special role (recall that, generically, we already have to shift all energy levels by a constant so that  $E_n \geq 0$ ), and therefore, the supersymmetric partner eigenstates exist for all  $n$ , including the ground state. Therefore, the Witten index, which is defined as the difference between the number of bosonic and fermionic ground states, is zero.

*1D, antiperiodic boundary conditions.*—Local Hamiltonians of the above type but with antiperiodic boundary conditions commute with the twisted translation operator  $\tilde{T}$ , which has the action

$$\hat{\tilde{T}}\gamma_i\hat{\tilde{T}}^{-1} = \gamma_{i+1} \quad (i < N), \quad (10)$$

$$\hat{\tilde{T}}\gamma_N\hat{\tilde{T}}^{-1} = -\gamma_1. \quad (11)$$

Since  $\hat{\tilde{T}}$  commutes with  $\hat{P}$ , the degeneracy found above is not required here.

However, we now show that for such Hamiltonians with antiperiodic boundary conditions, it is not possible for there to be both a unique ground state and a finite excitation gap in the thermodynamic limit. Such constraints, with origins in the Lieb-Schultz-Mattis theorem, have been recently established [41,42] for spin chains in which each unit cell transforms under a projective representation of a global symmetry (e.g., time reversal). We will now make contact

with these recent results by doubling the Majorana system and reinterpreting it as a spin system with additional symmetry from the doubling construction.

Assume for the sake of contradiction that the Hamiltonian  $H$  has a unique gapped ground state  $|\psi_0\rangle$ . Consider the doubled system  $H_D = H + \bar{H}$ , where  $\bar{H}$  is simply a second copy of  $H$  with Majorana operators represented by  $\bar{\gamma}$ . Since each subsystem has a unique gapped ground state, the composite  $H_D$  also has a unique gapped ground state  $|\psi_0\rangle \otimes |\psi_0\rangle$ . We now Jordan-Wigner transform  $H_D$  into a spin system:

$$\gamma_i = \left( \prod_{j<i} \sigma_j^z \right) \sigma_i^x, \quad (12)$$

$$\bar{\gamma}_i = \left( \prod_{j<i} \sigma_j^z \right) \sigma_i^y. \quad (13)$$

Care must be taken because the spin system with fixed boundary conditions only corresponds to a fixed fermion parity sector of the fermionic system. It is straightforward to check that the spin system with periodic boundary conditions corresponds to the fermion system with anti-periodic boundary conditions and even fermion parity. This sector includes the composite ground state  $|\psi_0\rangle \otimes |\psi_0\rangle$  (whose parity is the square of the parity of  $|\psi_0\rangle$ ).

Through the doubling construction,  $H_D$  has a set of discrete symmetries involving swapping the two chains. There is a  $Z_2$  group generated by  $\gamma \leftrightarrow \bar{\gamma}$  and a  $Z_4$  group generated by  $\gamma \rightarrow -\bar{\gamma}, \bar{\gamma} \rightarrow \gamma$ . In the spin language, these correspond, respectively, to the symmetries

$$\sigma_i^x \leftrightarrow (-1)^{i+1} \sigma_i^y, \quad (14)$$

$$\sigma_i^z \rightarrow -\sigma_i^z, \quad (15)$$

and

$$\sigma_i^x \rightarrow -\sigma_i^y, \quad (16)$$

$$\sigma_i^y \rightarrow \sigma_i^x, \quad (17)$$

$$\sigma_i^z \rightarrow \sigma_i^z. \quad (18)$$

Altogether, we have an on-site  $D_4$  group of rotations which is represented projectively (by spin 1/2).

Hence, the arguments in Refs. [41–43] rule out a gapped unique ground state of the spin system; in brief, a gapped unique ground state in one dimension is short-range entangled, and this local structure leads to incompatibility between the projective representation of each unit cell and translation symmetry. By contradiction, the original fermion chain cannot have a unique ground state with a gap in the thermodynamic limit.

*Two and higher dimensions.*—In this section, we first consider two-dimensional systems with translation symmetry in both directions, periodic boundary conditions, and a single Majorana mode per unit cell. If the system has one

odd length, then degeneracy of all energy levels follows by bundling all Majorana modes along the odd length direction into a supercell and applying the 1D argument above. However, this method does not apply to systems with two even dimensions, but nevertheless, the degeneracy holds. The fundamental reason is the fact, established below, that the two translations  $T_X$  and  $T_Y$  along the two directions  $X$  and  $Y$  anticommute when both dimensions  $N_X$  and  $N_Y$  are even. This implies (in conjunction with  $[\hat{T}_X, H] = [\hat{T}_Y, H] = 0$ ) that all states have at least twofold degeneracy.

We label the array of Majorana modes  $\gamma_{i,j}$  by their row  $i$  and column  $j$  positions. Translation  $T_X$  has projective representation given by the product of translations for each row:

$$\hat{T}_X = \prod_{r=1}^{N_Y} \hat{T}_{X,r}, \quad (19)$$

$$\hat{T}_{X,r} = \gamma_{r,1} \exp \left[ \frac{1}{4} \sum_{i,j=1}^{N_X} B_{ij} \gamma_{r,i} \gamma_{r,j} \right]. \quad (20)$$

See the Supplemental Material [48] for an explanation of why the translation operator has the above form; the essential feature is the Majorana operator  $\gamma_{r,1}$  ( $B$ , an antisymmetric matrix, is not important for our purposes). Then

$$\hat{T}_Y \hat{T}_X \hat{T}_Y^{-1} = \prod_{r=1}^{N_Y} \hat{T}_Y \hat{T}_{X,r} \hat{T}_Y^{-1} = \prod_{r=1}^{N_Y} \hat{T}_{X,r+1(\text{mod } N_Y)} \quad (21)$$

$$= -\hat{T}_X, \quad (22)$$

because each  $\hat{T}_{X,r}$  involves an odd number of distinct Majorana operators and there are an odd number ( $N_Y - 1$ ) of anticommutations required to return to the original ordering of  $\hat{T}_X$ . Thus,  $\{\hat{T}_X, \hat{T}_Y\} = 0$ , which ensures all states have at least twofold degeneracy. We note that in this case, the degeneracy is not due to supersymmetry [49].

The above results for periodic boundary conditions readily generalize to three-dimensional systems with at least one dimension of odd length, but we note that systems with three even length dimensions need not be degenerate. As a simple counterexample, consider a  $2 \times 2 \times 2$  array of Majorana modes with four-Majorana interactions on each face; this Hamiltonian has a unique ground state.

*Applications and phenomenology.*—Such Hamiltonians involving interacting Majorana modes serve as effective models for either the boundaries or vortex lattices of topological superconductors. For example, a stack of topological superconducting wires hosts an array of Majorana modes localized at the ends of the wires (see Fig. 2). The low energy physics of such systems is thus described by the interactions of the Majorana modes, for which our work is relevant.

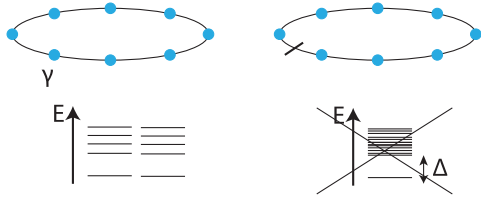


FIG. 1. Left: Translationally invariant Majorana modes, with periodic boundary conditions, have at least twofold degeneracy in the energy spectrum. The underlying supersymmetry requires that each energy level contains pairs of fermionic and bosonic superpartners. Right: The same system, with antiperiodic boundary conditions (depicted by a slash through a bond), cannot have a unique gapped ground state in the thermodynamic limit.

The particular interactions between Majorana modes depend on how the wires are coupled to each other, and as an example, a natural effective Hamiltonian for such systems is considered in Refs. [31,50]:

$$H = -it \sum_j \gamma_j \gamma_{j+1} + g \sum_j \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3}. \quad (23)$$

For a particular ratio of  $t/g$ , the above system is in the (supersymmetric) tricritical Ising universality class [31]. However, our work demonstrates that for *all* values of  $t$ ,  $g$ , the above system exhibits  $\mathcal{N} = 2$  supersymmetry, and as a consequence, all energy levels are at least doubly degenerate.

Such degeneracy between states of opposite fermion parity has a distinct signature in tunneling experiments: for a point contact located near the end point of one topological superconducting wire, there will be a zero-bias peak in the tunneling conductance. However, because the operator  $Q$  (which connects a state to its superpartner) is generically nonlocal, the zero-bias peak will be harder to observe as system size grows. More precisely, for a point contact to a noninteracting normal lead near the end point of wire  $j$ , the tunneling conductance is  $G \sim e^2/h$  if the temperature and the voltage bias are both smaller than  $\Lambda^* \propto |\langle 0|_F \langle \gamma_j | 0 \rangle_B|^2$ , where  $|0\rangle_F$  and  $|0\rangle_B$  are the degenerate ground-state

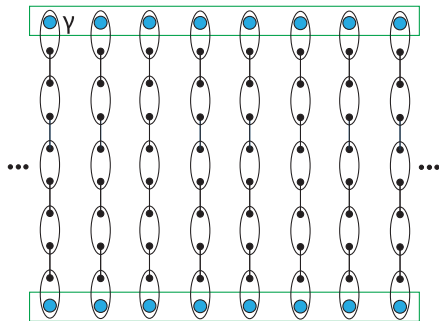


FIG. 2. Our work applies to effective Hamiltonians (green) describing the Majorana modes (blue) emerging at the end points of topological superconductors (vertical chains). Each oval is a spinless fermion consisting of two Majorana fermions (black dots).

superpartners [51,52]. For the Hamiltonian in Eq. (23), we find that  $|\langle 0|_F \langle \gamma_j | 0 \rangle_B|^2 = 2/N$  for any system size  $N$  in the noninteracting case of  $g = 0$ . Furthermore, by means of exact diagonalization, we verify that  $|\langle 0|_F \langle \gamma_j | 0 \rangle_B|^2 \propto N^\nu$ , with an exponent  $\nu = -1.0 \pm 0.1$  in the range of  $8 \leq N \leq 20$  for all parameter values  $|g/t| \leq 1$ . For relatively small numbers of superconducting wires (which are realistic for experiments), we therefore expect the zero-bias peak to be observable. Note that while the zero-bias peak is expected for a *single* Majorana mode, it is generically not present for an even number of modes; the supersymmetry is crucial here.

There are other routes toward experimental realization of the many models of the above type, including Abrikosov vortex lattices on the surface of topological insulators [53] and Josephson-coupled topological superconductor islands [54], in which charging energy mediates interactions between the emergent Majorana modes.

*Summary and discussion.*—We have shown that one- and two-dimensional systems of Majorana modes with translation symmetry, periodic boundary conditions, and odd number of modes per unit cell have at least twofold degeneracy for every state in the energy spectrum and that this is a reflection of an underlying  $\mathcal{N} = 2$  supersymmetry. Moreover, we have shown that such a one-dimensional system with antiperiodic boundary conditions cannot have a unique gapped ground state in the thermodynamic limit. Such Majorana systems may be realized at the boundaries or vortex lattices of topological superconductors, and the degeneracy arising from supersymmetry is potentially manifest as a zero-bias peak in tunneling experiments.

Our results motivate the conjecture that for all translationally invariant Majorana systems with an odd number of modes per unit cell, there cannot be a unique gapped ground state in the thermodynamic limit, regardless of dimension or boundary conditions. This leaves the possibilities of gaplessness and symmetry breaking in one dimension, and the additional possibility of topological order in higher dimensions. Furthermore, while our doubling analysis for antiperiodic boundary conditions in one dimensions conveniently makes use of recent spin system results, it would be very enlightening to find a direct proof of the result without having to double the system. It is not obvious to us how to apply the flux insertion arguments given by Oshikawa [55] and Hastings [56] in our case, since the symmetries are discrete.

Translation symmetry is only one of many crystal symmetries that can be considered. Other natural extensions include mirror reflection, inversion, and perhaps nonsymmorphic symmetries as well. The effect of these symmetries and their interplay with on-site symmetries such as time reversal is an intriguing direction for future work. For now, we note as a small extension of our work that mirror reflection and inversion each anticommute with fermion parity if the number of Majorana modes that are



transformed into other Majorana modes (and not themselves) is  $4n + 2$  for  $n \in \mathbb{Z}$ . We focused on translation symmetry because it enables the simplest manifestation of supersymmetry.

Finally, we note that the double degeneracy of the full spectrum discussed in our Letter can be thought of as ergodicity breaking that exists at all temperatures. This is because the degenerate eigenstates that differ in fermion parity cannot be connected by a local operator in the thermodynamic limit [note that the operator  $\hat{Q}$  in Eq. (6) is nonlocal in general]. Furthermore, if the system does not spontaneously break translational symmetry, finite energy density degenerate eigenstates  $|n\rangle_F$  and  $|n\rangle_B$  will satisfy  ${}_F\langle n|\hat{O}|n\rangle_F = {}_B\langle n|\hat{O}|n\rangle_B$  and  ${}_B\langle n|\hat{O}|n\rangle_F = 0$  for all local operators  $O$ , which is reminiscent of topological order [4,57].

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 [47] One could equally well choose  $\hat{Q}' = \sqrt{(H/2)}\hat{T}(\hat{1} - \hat{P})$  as the supersymmetric generator.  $\hat{Q}'$  and  $\hat{Q}$  are not independent because they are unitarily related:  $\hat{Q}' = \hat{T}^\dagger \hat{Q} \hat{T}$ .  
 [48] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.117.166802> for a

- review of the Majorana representation of  $O(N)$  and the explicit form of the translation operator.
- [49] However, one can consider a “screw” boundary condition in which translation  $T_Y$  across a boundary is supplemented by a translation in  $X$ ; the array is then effectively a one-dimensional chain in which the head of one column is identified with the tail of the *next* column. For this case,  $T_Y$  anticommutes with fermion parity and the system is thus supersymmetric.
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