AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET XI
(due Tuesday, August 3, 2021)

Problem 1

(a) We continue here with our discussion of the coupled harmonic oscillator from Lecture 11 where we showed that the equations of motion for the two masses were given by

$$F_n = m\,\ddot{u}_n = -2\alpha u_n + \alpha u_{n+1}$$

$$F_{n+1} = m\,\ddot{u}_{n+1} = -2\alpha u_{n+1} + \alpha u_n$$

If we assume solutions of the form for these two equations

$$u_n = A_n\, e^{i\omega t}$$

$$u_{n+1} = A_{n+1}\, e^{i\omega t}$$

where we will assume that ω is the frequency of our system. Show that you can obtain the following real equations

$$-m\omega^2 A_n = -2\alpha\, A_n + \alpha\, A_{n+1}$$

$$-m\omega^2 A_{n+1} = -2\alpha\, A_{n+1} + \alpha\, A_n.$$

(b) Show that we can recast these equations in the form of an eigenvalue problem

$$\boldsymbol{A\vec{v} = \lambda\vec{v}}$$

where

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{2\alpha} & \boldsymbol{-\alpha} \\ \boldsymbol{-\alpha} & \boldsymbol{2\alpha} \end{pmatrix}$$

$$\vec{\mathbf{v}} = \begin{pmatrix} \boldsymbol{A_n} \\ \boldsymbol{A_{n+1}} \end{pmatrix}$$

and

$$\boldsymbol{\lambda = m\omega^2.}$$

(c) Solve the corresponding characteristic equation to discover that our coupled harmonic oscillator problem now has **two** distinct eigenvalues or frequencies

$$\boldsymbol{\omega_1} = \sqrt{\frac{\boldsymbol{\alpha}}{\boldsymbol{m}}}$$

$$\boldsymbol{\omega_2} = \sqrt{\frac{\boldsymbol{3\alpha}}{\boldsymbol{m}}}$$

(d) For each eigenvalue or frequency solve for the corresponding eigenvector to yield

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{a} \end{pmatrix}$$

and

$$\vec{\mathbf{v}}_2 = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{-a} \end{pmatrix}$$

where **a** is an arbitrary constant.

(e) Now return to the original assumptions for our solutions in Part 1(a). Having solved our eigenvalue problem for the mass at position **n**

$$\boldsymbol{u_n = A_n\, e^{i\omega t}}$$

we discovered that there are **two** allowed frequencies thus giving two different solutions

$$\boldsymbol{u_n = A_n\, e^{i\omega_1 t} = a\, e^{i\omega_1 t}}$$

and

$$\boldsymbol{u_n = A_n\, e^{i\omega_2 t} = a\, e^{i\omega_2 t}}$$

We can express the general solution to our original differential equation as a linear combination of these two solutions

$$\boldsymbol{u_n = c_1 a\, e^{i\omega_1 t} + c_2 a\, e^{i\omega_2 t}}$$

where $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are arbitrary constants.

The same line of reasoning applies to the mass at position **n+1**

$$\boldsymbol{u_{n+1} = A_{n+1}\, e^{i\omega t}}$$

that there are **two** allowed frequencies again thus yielding two different solutions

$$\boldsymbol{u_{n+1} = A_{n+1}\, e^{i\omega_1 t} = a\, e^{i\omega_1 t}}$$

and

$$\boldsymbol{u_{n+1} = A_{n+1}\, e^{i\omega_2 t} = -a\, e^{i\omega_2 t}}$$

We can also express the general solution to our original differential equation as a linear combination of these two solutions

$$\boldsymbol{u_{n+1} = c_1 a\, e^{i\omega_1 t} - c_2 a\, e^{i\omega_2 t}}$$

where $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are arbitrary constants.

If we define new constants

$$\boldsymbol{b_1 = ac_1}$$

$$\boldsymbol{b_2 = ac_2}$$

we obtain two simplified equations

$$\boldsymbol{u_n = b_1\, e^{i\omega_1 t} + b_2\, e^{i\omega_2 t}}$$

and

$$\boldsymbol{u_{n+1} = b_1\, e^{i\omega_1 t} - b_2\, e^{i\omega_2 t}}$$

(f) Now what are these equations in Part 1(e) telling us? We will now show that for $\boldsymbol{b_2 = 0}$ and $\boldsymbol{b_1 \neq 0}$ our two masses will vibrate in phase. Let us introduce an arbitrary phase $\boldsymbol{\phi_1}$ to the problem such that

$$\boldsymbol{b_1 = A_1\, e^{-i\phi_1}}$$

Show that our two equations now become

$$\boldsymbol{u_n(t) = A_1 \cos\left(\omega_1 t - \phi_1\right)}$$

$$\boldsymbol{u_{n+1}(t) = A_1 \cos\left(\omega_1 t - \phi_1\right)}$$

which shows that the two masses vibrate in unison.

(g) We will next consider what our two equations in Problem 1(e) are telling us for the case where $\boldsymbol{b_1 = 0}$ and $\boldsymbol{b_2 \neq 0}$. Here our two masses will vibrate in opposite directions. Let us introduce an arbitrary phase $\boldsymbol{\phi_2}$ to the problem such that

$$\boldsymbol{b_2 = A_2\, e^{-i\phi_2}}$$

Show that our two equations from Problem 1(e) now become

$$\boldsymbol{u_n(t) = A_2 \cos\left(\omega_2 t - \phi_2\right)}$$

$$\boldsymbol{u_{n+1}(t) = -A_2 \cos\left(\omega_2 t - \phi_2\right) = A_2 \cos\left(\omega_2 t - \phi_2 + \pi\right)}$$

which means that our masses vibrate out of phase by 180^o and hence in opposite directions. Our final results from Problem 1(f) and Problem 1(g) tell us the **normal modes** of vibration for our coupled harmonic oscillator system. Of course the real system will be described by a superposition of these two normal modes of vibration. (*Cf.* Dr. H. Trevor Johnson-Steigelman, The University of Maryland at College Park, (http://youtu.be/JOSK7Zy8iko)

Problem 2

Diagonalize the following matrix **Q**:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \end{pmatrix}$$

Problem 3

Diagonalize the following matrix **Z**:

$$\mathbf{Z} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Problem 4

Diagonalize the following matrix **M**:

$$\mathbf{M} = \begin{pmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{6} \end{pmatrix}$$

Problem 5

Diagonalize the following matrix **B**:

$$\mathbf{B} = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 6

Diagonalize the following matrix **F**:

$$\mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Problem 7

Diagonalize the following matrix **K**:

$$\mathbf{K} = \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 8

Diagonalize the following matrix $\mathbf{J}$:

$$\mathbf{J} = \begin{pmatrix} \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 9

Diagonalize the following matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ -\mathbf{1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$$

Problem 10

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Linear Transformations and Matrices, Essence of Linear Algebra: Chapter 3 (10:58 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he shows how matrices are examples of things called **"linear transformations"** in a very clear and geometric way! I highly suggest looking at it once you have mastered Lecture 9 and Problem Set 9.

Python Exercise 11

1. The following is a Python script (a fancy word for a program) for diagonalizing a matrix. Note the similarity with the Python script for finding the eigenvalues and eigenvectors of a matrix.

import numpy as np

A = np.array([[1,2],[2,1]])#defining the matrix A

print(A)#print the original matrix A

L,V =np.linalg.eig(A)# Note that the function **np.linalg.eig()** returns the eigenvalues in a vector and the eigenvectors in a matrix. Here L stands for the eigenvalues (Lambda) and V stands for the eigenvectors (Vectors)

print(L)#print the eigenvalues of A

print(V)#print the eigenvectors of A

D = np.diag(L)#Note that the function **np.diag()** takes the eigenvalues L and forms a diagonal matrix **D**.

print(D)#print this diagonal matrix D generated from above

2. Create the following random matrices of sizes: **(5 x 5)**, **(10 x 10)**, **(20 x 20)**, and **(50 x 50)**. Use Python to find their eigenvalues and eigenvectors. In the above script the diagonal matrix is simply built from the eigenvalues. Modify this script to explicitly compute the diagonal matrix **D** using the expression

$$\boldsymbol{D} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$$

3. In the beginning of this course we introduced the concept of the **rank** of a matrix to provide a neat way of determining if a set of linear equations had a unique solution, an infinite number of solutions, or no solution. This definition of the **rank** of a matrix involved calculating determinants which was something we gained expertise in. Actually it turns out that this is just one definition for the **rank** of a matrix and the concept of the **rank** of a matrix is very powerful and it has many definitions.

Perhaps the most important definition of the **rank** of a matrix is the maximum number of column vectors it has which are linearly independent. Obviously we could not use this definition earlier in the course as we did not know what linear independence meant! Well that was then and this is now.

Calculating the **rank** of a large matrix is quite an involved thing. Here we see how Python does it using the **np.linalg.matrix _rank()** function

import numpy as np

P = np.array([[1,2,],[2,1]])

print(P)

r1 = np.linalg.matrix_rank(P)

print(r1)

Go back to Problem Set V (Problems 1, 2, and 3) and verify your results for both the coefficient and augmented matrices.