AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET X
(due Tuesday, July 27, 2021)

Problem 1

Suppose we have a new basis $\vec{\boldsymbol{B}}'$ where the basis vectors are given by

$$\hat{e}_1' = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\hat{e}_2' = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

$$\hat{e}_3' = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

and suppose we have a vector

$$\vec{\boldsymbol{v}}_{\vec{\boldsymbol{B}}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

where of course $\vec{\boldsymbol{B}}$ is the standard basis. Find the change in basis matrix for the problem and find $\vec{\boldsymbol{v}}_{\vec{\boldsymbol{B}}'}$. Explicitly check that the inverse of the change in basis matrix does what it is supposed to do correctly.

Problem 2

Suppose we have a new basis $\vec{\boldsymbol{B}}'$ where the basis vectors are given by

$$\hat{e}'_{\mathbf{1}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \end{pmatrix}$$

$$\hat{e}'_{\mathbf{2}} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{4} \end{pmatrix}$$

and suppose we have a vector

$$\vec{\boldsymbol{v}}_{\boldsymbol{B}} = \begin{pmatrix} \mathbf{4} \\ \mathbf{6} \end{pmatrix}$$

which is expressed in the standard basis. Find the change in basis matrix that will express $\vec{\boldsymbol{v}}_{\vec{\boldsymbol{B}}}$ in terms of your new basis. Explicitly check that the inverse of the change in basis matrix does what it is supposed to do correctly.

Problem 3

Find the eigenvalues and eigenvectors of the following matrix **Z**:

$$\mathrm{Z} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} + \boldsymbol{\lambda_3} = \boldsymbol{Tr}\,\mathbf{Z}$ and $\mathbf{det}\ \mathbf{Z} = \boldsymbol{\lambda_1 \lambda_2 \lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of **Z**. Recall from Problem Set III that Tr **Z** is the trace of the matrix **Z** which is the sum of all the diagonal elements of a matrix.

Problem 4

Find the eigenvalues and eigenvectors of the following matrix **M**:

$$\mathrm{M} = \begin{pmatrix} \mathbf{-1} & \mathbf{4} \\ \mathbf{-2} & \mathbf{5} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} = \boldsymbol{Tr}\,\mathbf{M}$ and $\mathbf{det}\ \mathbf{M} = \boldsymbol{\lambda_1 \lambda_2}$ where λ_1 and λ_2 are the eigenvalues of **M**.

Problem 5

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} = \boldsymbol{Tr}\,\mathbf{P}$ and $\mathbf{det}\ \mathbf{P} = \boldsymbol{\lambda_1 \lambda_2}$ where λ_1 and λ_2 are the eigenvalues of $\mathbf{P}$.

Problem 6

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{Y}$:

$$\mathbf{Y} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} + \boldsymbol{\lambda_3} = \boldsymbol{Tr}\,\mathbf{Y}$ and $\mathbf{det}\ \mathbf{Y} = \boldsymbol{\lambda_1 \lambda_2 \lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{Y}$.

Problem 7

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} = \boldsymbol{Tr}\,\mathbf{D}$ and $\mathbf{det}\ \mathbf{D} = \boldsymbol{\lambda_1 \lambda_2}$ where λ_1 and λ_2 are the eigenvalues of $\mathbf{D}$.

Problem 8

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} \mathbf{3} & \mathbf{-2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-4} & \mathbf{-3} \\ \mathbf{2} & \mathbf{-4} & \mathbf{0} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} + \boldsymbol{\lambda_3} = \boldsymbol{Tr}\,\mathbf{T}$ and $\mathbf{det}\ \mathbf{T} = \boldsymbol{\lambda_1 \lambda_2 \lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{T}$.

Problem 9

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Change of Basis, Essence of Linear Algebra: Chapter 13 (12:50 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he shows how the change of basis matrix works in a very clear and geometric way! I highly suggest looking at it once you have mastered Lecture 10 and Problem Set 10.

Problem 10

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Eigenvectors and Eigenvalues, Essence of Linear Algebra: Chapter 14 (just look at the first 13:04 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he describes eigenvectors and eigenvalues in a very clear and geometric way! His discussion of the determinant using geometry is interesting. You can either skip it, as it is very brief, or look at his Lecture 6 if you want. I highly suggest looking at his Lecture 14 once you have mastered our Lecture 10 and Problem Set 10.

Python Exercise 10

1. The following is a Python script (a fancy word for a program) for finding the eigenvalues and eigenvectors of a matrix

```
import numpy as np
A = np.array([[1,2],[2,1]])
print(A)
L,V =np.linalg.eig(A)# Here L stands for the eigenvalues (Lambda) and V stands for the eigenvectors (Vectors)
print(L)
print(V)
```

Note that the function **np.linalg.eig()** returns the eigenvalues in a vector and the eigenvectors in a matrix. Also note that the eigenvectors are normalized. Use this script to verify your results in Problems 3-8 of this problem set.

2. Create the following random matrices of sizes: **(5 x 5)**, **(10 x 10)**, **(20 x 20)**, and **(50 x 50)**. Use Python to find their eigenvalues and eigenvectors. See if your matrices are invertible.

3. The $\boldsymbol{n^{th}}$ power of a square matrix is denoted by $\boldsymbol{A^n}$. Why won't this operation work for a rectangular matrix? In Python this power is formed using the function **np.linalg.matrix_power(A,2)** where n = 2, for example.

```
import numpy as np
A = np.array([[1,5],[3,4]])
print(A)
T =np.linalg.matrix_power(A,2)# evaluate the matrix A raised to the power of 2
print(A)
```

Experiment with this Python script for matrices where the matrix elements are all larger than 1 and all less than one. Do you see any trends? We will come back to this idea in Problem Set XII.