AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET VIII
(due Tuesday, July 13, 2021)

Problem 1

Find the linear combination

$$2\vec{\mathbf{v}}_1 + 3\vec{\mathbf{v}}_2 + 4\vec{\mathbf{v}}_3 = \vec{\mathbf{b}}$$

for the following vectors

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and simplify your results. Next, write $\vec{\boldsymbol{b}}$ as a matrix-vector multiplication $\mathbf{A}\vec{\boldsymbol{x}}$ where the matrix $\mathbf{A}$ has columns which are built from $\vec{\boldsymbol{v}}_1$,$\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ and $\vec{\boldsymbol{x}}$ is the matrix containing the coefficients expressed in the linear combination. Show that this result is identical to your previous result.

Problem 2

Find a linear combination

$$x_1\vec{\mathbf{w}}_1 + x_2\vec{\mathbf{w}}_2 + x_3\vec{\mathbf{w}}_3 = \vec{\mathbf{0}}$$

for the following vectors

$$\vec{\mathbf{w}}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

$$\vec{\mathbf{w}}_2 = \begin{pmatrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{pmatrix}$$

$$\vec{\mathbf{w}}_3 = \begin{pmatrix} \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{pmatrix}$$

Are the vectors $\vec{\boldsymbol{w}}_1$, $\vec{\boldsymbol{w}}_2$, and $\vec{\boldsymbol{w}}_3$ linearly dependent or linearly independent?. Do these three vectors lie in a line, plane, or in three-dimensions? Consider the matrix $\mathbf{W}$ which has columns which are built from $\vec{\boldsymbol{w}}_1$, $\vec{\boldsymbol{w}}_2$, and $\vec{\boldsymbol{w}}_3$. Is $\mathbf{W}$ invertible?

Problem 3

Find a linear combination

$$f_1\vec{\mathbf{c}}_1 + f_2\vec{\mathbf{c}}_2 + f_3\vec{\mathbf{c}}_3 = \vec{\mathbf{0}}$$

for the following vectors

$$\vec{\mathbf{c}}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{7} \end{pmatrix}$$

$$\vec{\mathbf{c}}_2 = \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{8} \end{pmatrix}$$

$$\vec{\mathbf{c}}_3 = \begin{pmatrix} \mathbf{3} \\ \mathbf{6} \\ \mathbf{9} \end{pmatrix}$$

Are the vectors $\vec{\boldsymbol{c}}_1$, $\vec{\boldsymbol{c}}_2$, and $\vec{\boldsymbol{c}}_3$ linearly dependent or linearly independent?. Do these three vectors lie in a line, plane, or in three-dimensions? Consider the matrix $\mathbf{C}$ which has columns which are built from $\vec{\boldsymbol{c}}_1$, $\vec{\boldsymbol{c}}_2$, and $\vec{\boldsymbol{c}}_3$. Is $\mathbf{C}$ invertible?

Problem 4

What value must **c** be to ensure that the column vectors of the following matrix are linearly dependent?

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{1} & \boldsymbol{c} \end{pmatrix}$$

Problem 5

Consider a cube where four of its corners are given by the vectors

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

What are the other four corners? Find the coordinates of the center point of the cube. What are the center points of the six faces? Using this problem as a hint how many corners does a cube have in **four** dimensions?

Problem 6

Describe geometrically (line, plane or all of $\boldsymbol{R}^{\mathbf{3}}$) all linear combinations of the following sets of vectors:

(a)

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{3} \\ \mathbf{6} \\ \mathbf{9} \end{pmatrix}$$

(b)

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

(c)

$$\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

Problem 7

Given the following complex vectors

$$\vec{u} = \begin{pmatrix} 1+i \\ 3 \\ 4-i \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 3-4i \\ 1+i \\ 2i \end{pmatrix}$$

compute the following

$$(i) \quad \langle \vec{u} | \vec{v} \rangle$$

$$(ii) \quad \langle \vec{v} | \vec{u} \rangle$$

$$(iii) \quad \|\vec{u}\|$$

$$(iv) \quad \|\vec{v}\|$$

Problem 8

Show that the following set of vectors form an orthogonal set. How would you make them orthonormal?

$$\vec{u} = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 0 \\ 1-i \\ i \end{pmatrix}$$

$$\vec{w} = \begin{pmatrix} 3i-3 \\ 1+i \\ 2 \end{pmatrix}$$

Problem 9

Show that the following three vectors are not linearly independent by expressing one of them as a linear combination of the other

$$\vec{\boldsymbol{u}}_1 = \begin{pmatrix} \mathbf{1} \\ \boldsymbol{i} \\ \mathbf{-1} \end{pmatrix}$$

$$\vec{\boldsymbol{u}}_2 = \begin{pmatrix} \mathbf{1} + \boldsymbol{i} \\ \mathbf{0} \\ \mathbf{1} - \boldsymbol{i} \end{pmatrix}$$

$$\vec{\boldsymbol{u}}_3 = \begin{pmatrix} \boldsymbol{i} \\ \mathbf{-1} \\ -\boldsymbol{i} \end{pmatrix}$$

Problem 10

Determine if the following vectors are linearly independent

$$\vec{\boldsymbol{u}}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ -\boldsymbol{i} \end{pmatrix}$$

$$\vec{\boldsymbol{u}}_2 = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{i} \\ \boldsymbol{i} \end{pmatrix}$$

$$\vec{\boldsymbol{u}}_3 = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$$

Problem 11

Determine if the following vectors are linearly independent

$$\vec{u}_1 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} i \\ i \\ 0 \end{pmatrix}$$

$$\vec{u}_3 = \begin{pmatrix} i \\ i \\ i \end{pmatrix}$$

Problem 12

Use Euler's formula to prove de Moivre's formula where **z** is a complex number

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Use this result to show that

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

and

$$\sin 2\theta = 2 \cos \theta \sin \theta.$$

(Hint: Consider de Moivre's formula for the case where $\mathbf{n} = 2$.

Problem 13

The following matrices are called the **Pauli spin matrices** and they are very important in quantum mechanics and quantum computing

$$\mathbf{I} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{\mathbf{x}} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{\mathbf{y}} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{\mathbf{z}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

Show that the Pauli spin matrices are linearly independent.

Problem 14

Recall that for two complex vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ the inner product is given by

$$\langle \vec{\boldsymbol{u}} | \vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{v}} | \vec{\boldsymbol{u}} \rangle^*$$

Verify this identity explicitly for the following two vectors:

$$\vec{\boldsymbol{u}} = \begin{pmatrix} \mathbf{1} + \boldsymbol{i} \\ \mathbf{1} \end{pmatrix}$$

$$\vec{\boldsymbol{v}} = \begin{pmatrix} -\boldsymbol{i} \\ -\mathbf{1} \end{pmatrix}$$

Problem 15

Recall that for two complex vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ the inner product is given by

$$\langle \vec{\boldsymbol{u}} | \vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{v}} | \vec{\boldsymbol{u}} \rangle^*$$

Verify this identity explicitly for the following two vectors:

$$\vec{\boldsymbol{u}} = \begin{pmatrix} \mathbf{3} \\ -\boldsymbol{i} \\ \mathbf{2}\boldsymbol{i} \end{pmatrix}$$

$$\vec{\boldsymbol{v}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{3}\boldsymbol{i} \\ -\mathbf{1} \end{pmatrix}$$

Python Exercise 8

1. Let us now see how to handle complex numbers in Python:

import numpy as np

z = np.complex(3,4) # Here we define the variable z as the complex number 3 + 4j. Note that Python uses j instead of i for complex numbers which is in agreement with the convention of electrical engineers

print(z)

Define the complex number r as follows

r = 3 +4j

Now define another complex number and add it to r and print your answer. Next try multiplying these two numbers together.

2. Let us now explore the complex conjugate operation in NumPy:

R = np.random.randint(-3, 4, size=3)# Generate a random 1 x 3 matrix of random integers between -3 and 4

print (R)

I = np.random.randint(-3, 4, size=3)# Generate a random 1 x 3 matrix of random integers between -3 and 4

print (I)

Z = R + I *1j # Given R as the matrix containing the real parts of a complex number and I as the matrix containing the imaginary parts of a complex number, create the matrix Z of complex numbers

print (Z)

print(Z.conj())# Using the function conj() for the complex conjugate, print the complex conjugate of the matrix Z

Using Python revisit Problem 7 parts (iii) and (iv) with the two complex vectors in the problem. Note that the square root function in Python is given by **sqrt()**. Repeat this experiment with two random complex vectors containing **ten** components.

3. Let us now explore the dot product operation of two complex vectors in NumPy:

import numpy as np

v = [2,1j] # Here we build the complex vector 2 + j

print(np.vdot(v,v))# Use the function vdot() to take the dot product of two complex vectors

Using Python revisit Problem 7 parts (i) and (ii) with the two complex vectors in the problem. Repeat this experiment with two random complex vectors containing **ten** components.

4. A Hermitian matrix is the complex-valued equivalent of something between a **symmetric matrix**

$$\mathbf{A} = \mathbf{A^T}$$

and a **skew-symmetric** matrix

$$\mathbf{A} = -\mathbf{A^T}.$$

A Hermitian matrix obeys the following equation

$$\mathbf{A} = (\mathbf{A}^*)^\mathbf{T} = (\mathbf{A^T})^* = (\mathbf{A})^\dagger$$

where the dagger is shorthand for either taking the complex conjugate first and then the transpose or the transpose first and then the complex conjugate. The dagger operation is referred to as the **Hermitian conjugate**.

Let us refer to Python Exercise 6 of Problem Set V where you discovered how to make a random 3 x 3 matrix symmetric. There is another way to make a square matrix symmetric and it is listed below for you to prove

$$\frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{A} + \mathbf{A^T}\right)$$

Using your knowledge of Python thus far, write a program to generate two random real 3 x 3 matrices and then use these results to build a complex matrix **A**. Use both of these two previously discussed approaches to generate two new matrices and test whether your new matrices are either symmetric, Hermitian, or neither. Note that the Python command to find the Hermitian conjugate of the matrix **A** is **A.H**. You may also find the following Python command useful $\boldsymbol{A = np.matrix(R + I * 1j)}$ where you can guess what **R** and **I** mean.