AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET VII
(due Tuesday, July 6, 2021)

Problem 1

The following table shows the test scores of seven students on three tests:

Test Scores of Seven Students			
Student's Last Name	Test 1	Test 2	Test 3
Jones	90	75	60
Chan	54	92	70
Rocco	63	70	81
Johnson	70	71	72
Stein	46	90	63
Rio	87	72	69
Smith	50	77	83

View the columns in the body of the table as vectors $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$, in $\boldsymbol{R}^7$, and view the rows in the body of the table as rows $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_4$, $\vec{r}_5$, $\vec{r}_6$, and $\vec{r}_7$, in $\boldsymbol{R}^3$.

(i) Find the scalars k_1, k_2, and k_3 such that the components of the vector

$$\vec{x} = k_1\vec{c}_1 + k_2\vec{c}_2 + k_3\vec{c}_3$$

are the average test scores for the students.

(ii) Find the scalars k_1, k_2, k_3 k_4, k_5, k_6, k_7 such that the components of the vector

$$\vec{x} = k_1\vec{r}_1 + k_2\vec{r}_2 + k_3\vec{r}_3 + k_4\vec{r}_4 + k_5\vec{r}_5 + k_6\vec{r}_6 + k_7\vec{r}_7$$

are the average scores for all of the students on each test.

(iii) Give an interpretation for the vector

$$\vec{x} = \tfrac{1}{4}\vec{c}_1 + \tfrac{1}{4}\vec{c}_2 + \tfrac{1}{2}\vec{c}_3$$

Problem 2

The following table shows the population of five Pennsylvania counties in four different years:

Population of Five Pennsylvania Counties				
	1950	1980	1992	1998
Philadelphia	408,762	847,170	1,552,572	1,436,287
Bucks	144,620	479,211	556,279	587,942
Delaware	414,234	555,007	549,506	542,593
Adams	44,197	68,292	81,232	86,537
Potter	16,810	17,726	16,863	17,184

View the columns in the body of the table as vectors $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$,and $\vec{c}_4$, in R^5, and view the rows in the body of the table as rows $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_4$, and $\vec{r}_5$, in R^4.

(i) Find the scalars k_1, k_2, k_3, and k_4 such that the components of the vector

$$\vec{x} = k_1\vec{c}_1 + k_2\vec{c}_2 + k_3\vec{c}_3 + k_4\vec{c}_4$$

are the average populations of the counties over the four sampled years.

(ii) Find the scalars k_1, k_2, k_3 k_4, and k_5 such that the components of the vector

$$\vec{x} = k_1\vec{r}_1 + k_2\vec{r}_2 + k_3\vec{r}_3 + k_4\vec{r}_4 + k_5\vec{r}_5$$

are the average populations of the five counties in each sampled year.

(iii) Give an interpretation for the vector

$$\vec{x} = \tfrac{1}{3}\vec{c}_1 + \tfrac{1}{3}\vec{c}_2 + \tfrac{1}{3}\vec{c}_3$$

Problem 3

Find the angle θ between a diagonal of a cube and one of its edges.

Problem 4

What can you say about the angle between the nonzero vectors $\vec{u}$ and $\vec{v}$ in $\boldsymbol{R}^2$ if

$$\vec{u} \cdot \vec{v} > 0?$$

What if

$$\vec{u} \cdot \vec{v} < 0?$$

Problem 5

Many books published in the last 40 years have been assigned a unique 10-digit number called an **International Standard Book Number** or ISBN. The first nine digits of this number are split into three groups - the first group representing the country or group of countries in which the book originates, the second identifying the publisher, and the third assigned to the book title itself. The tenth and final digit, called a **check digit,** is computed from the first nine digits and is used to ensure that an electronic transmission of the ISBN, say over the Internet, occurs without error.

To explain how this is done, regard the first nine digits of the ISBN as a vector $\vec{b}$ in $\mathbf{R}^9$ and let $\vec{a}$ be the vector

$$\vec{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

Then the check digit c is computed using the following procedure. First form the dot product $\vec{a} \cdot \vec{b}$. Next divide this dot product by 11, thereby producing a remainder c that is an integer between 0 and 10, inclusive. The check digit is taken to be c with the proviso that c = 10 is written as X to avoid double digits.

For example, the ISBN of the brief edition of *Calculus*, sixth edition, by Howard Anton is 0-471-15307-9 which has a check digit of 9. This is consistent with the first nine digits of the ISBN, since $\vec{a} \cdot \vec{b} = 152$.

Dividing 152 by 11 produces a quotient of 13 and a remainder of 9, so the check digit is c = 9. If an electronic order is placed for a book with a certain ISBN, then the warehouse can use the above procedure to verify that the check digit is consistent with the first nine digits, thereby reducing the possibility of a costly shipping error. Please verify whether **1-56592-170-7** is a valid ISBN by computing its check digit.

Problem 6

Complete the sentences: The intersection of two planes through

$$(0, 0, 0)$$

is probably ??? but it could be a ??? . It can not be a ???.

Problem 7

What geometric property must a set of two vectors have if they are to span $\boldsymbol{R}^2$?

Problem 8

What geometric property must a set of three vectors in $\boldsymbol{R}^3$ have if they are to span $\boldsymbol{R}^3$?

Problem 9

Given the vectors

$$\vec{\mathbf{v}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

$$\vec{\mathbf{w}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{5} \end{pmatrix}$$

choose a number **c** such that $\vec{\boldsymbol{w}} - \mathbf{c}\vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{v}}$. Then find the formula that gives this number **c** for any nonzero $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$. Note that $\mathbf{c}\vec{\boldsymbol{v}}$ is the **projection** of $\vec{\boldsymbol{w}}$ onto $\vec{\boldsymbol{v}}$. We will see this idea again in Lecture 12 but it does not hurt to look at it now!

Problem 10

How long is the vector

$$\vec{\mathbf{v}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

in 9 dimensions? Find a unit vector $\hat{\boldsymbol{u}}$ in the same direction as $\vec{\boldsymbol{v}}$ and a unit vector $\hat{\boldsymbol{w}}$ that is perpendicular to $\vec{\boldsymbol{v}}$.

Python Exercise 7

1. Please verify whether **0-471-05333-5** is a valid ISBN by computing its check digit. Make your life easy and use the appropriate Python commands.

2. Please verify whether **0-471-06368-1** is a valid ISBN by computing its check digit. Make your life easy and use the appropriate Python commands.

3. Please verify whether **0-13-947752-3** is a valid ISBN by computing its check digit. Make your life easy and use the appropriate Python commands.

4. Given the matrix equation

$$\vec{\boldsymbol{A}}\,\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$$

use Python to generate a random matrix $\vec{\boldsymbol{A}}$ (50 x 50) matrix and a random matrix $\vec{\boldsymbol{b}}$ (50 x 1). Solve the problem for $\vec{\boldsymbol{x}}$ (50 x 1) by finding the inverse of $\vec{\boldsymbol{A}}$ (50 x 50). Now solve the problem using Gauss-Jordan elimination in Python. Using the **time** command in Python which approach is faster?

5. In coding one needs to be efficient and not repeat algorithms that do the same thing many, many times in the code.One way to effect this goal is to use **loops** or in this particular case, the **while** loop in Python. For example, suppose you want to code the following **infinite sum** in Python

$$\boldsymbol{1 + x + x^2 + x^3 + \cdots}$$

where x is any real number. The way to do this is to on a computer is to find a finite sum $\mathbf{S_n(x)}$

$$\boldsymbol{S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n}$$

and explore the limit of the finite sum as n approaches infinity.

Here is the Python code to compute a partial sum $\boldsymbol{S_n(x)}$, where the infinite sum is known as the **geometric series**:

x = 0.50 # This is where we define the variable x

n = 0 #Let us define the counter n as 0

partialsum = 0 #Let us start with the partial sum or the variable partialsum as equal to 0

while n <= 100: # Here is where we define a **while** loop where everything after the colon : is performed up until the point that n = 99 < 100. Please note that all the actions you wish the loop to perform must be indented in Python

partialsum = partialsum + x**n #Here is where we start with partialsum = 0 and then add x^0 to get a new value for partialsum. Please note than ** stands for taking a power.

n += 1 # Now increase the counter n by 1.

print(partialsum) Now print out the value of partial sum and return to the **while** command and go through the next loop for n =1. You keep repeating this loop until n = 99.

Now can you see what the limit of this partial sum is as n gets larger for x = 0.5? Repeat this for x = 0.1, 0.4. 0.9, 1.2, 4, 20, 100, and finally 1.0. Can you see any interesting results here from your computer experiments? You can also play with the cutoff 100 in the **while** statement.

6. Using the Python example for a **while** loop in the previous problem, experiment with the following infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Test for the cases where p > 1, p < 1, and p =1. What do your experiments tell you about this infinite series known as the **p-series**?

7. Using the Python example for a **while** loop in Problem 5, experiment with the following infinite series.

$$\sum_{n=1}^{\infty} \frac{x^n}{n\,2^n}$$

Test for the cases where x > 2, x < 2, and x =2. What do your experiments tell you about this infinite series which is an example of a power series?

Problem 1

(i) For example

$$\vec{C}_1 = (90, 54, 63, 70, 46, 87, 50)$$ (all student scores on Test 1)

$$\vec{C}_2 = (75, 92, 70, 71, 90, 72, 77)$$

(all student scores on Test 2)

$\vdots$

$$\vec{C}_3 = (60, 70, 81, 72, 63, 69, 83)$$

or as matrices

$$\vec{C}_1 = \begin{pmatrix} 90 \\ 54 \\ \vdots \\ \vdots \end{pmatrix} \qquad \vec{C}_2 = \begin{pmatrix} 75 \\ 92 \\ \vdots \\ \vdots \end{pmatrix}$$

$$\vec{C}_3 = \begin{pmatrix} 60 \\ 70 \\ \vdots \\ \vdots \end{pmatrix}$$

$$X_1 = k_1 90 + k_2 75 + k_3 60$$

If X_1 is the average test score for Jones then

$$X_1 = \frac{1}{3} 90 + \frac{1}{3} 75 + \frac{1}{3} 60$$

so

$$\vec{X} = \underbrace{\frac{1}{3}}_{k_1} \vec{C}_1 + \underbrace{\frac{1}{3}}_{k_2} \vec{C}_2 + \underbrace{\frac{1}{3}}_{k_3} \vec{C}_3$$

(ii) For example

$\vec{r}_1 = (90, 75, 60)$ ← all test scores for Jones

$\vec{r}_2 = (54, 92, 70)$ ← all test scores for Chao

$\vdots$

$\vec{r}_7 = (50, 77, 83)$ ← all test scores for Smith

or as matrices

$$\vec{r}_1 = \begin{pmatrix} 90 \\ 75 \\ 60 \end{pmatrix} \quad \vec{r}_2 = \begin{pmatrix} 54 \\ 92 \\ 70 \end{pmatrix} \quad \cdots \quad \vec{r}_7 = \begin{pmatrix} 50 \\ 77 \\ 83 \end{pmatrix}$$

$$x_1 = k_1\,90 + k_2\,54 + \cdots k_7\,50$$

If x_1 is the average of all of the students performances on Test 1

then $x_1 = \frac{1}{7}\,90 + \frac{1}{7}\,54 + \cdots \frac{1}{7}\,50$

so

$$\vec{x} = \underbrace{\tfrac{1}{7}}_{k_1}\vec{r}_1 + \underbrace{\tfrac{1}{7}}_{k_2}\vec{r}_2 + \underbrace{\tfrac{1}{7}}_{k_3}\vec{r}_3 + \cdots \underbrace{\tfrac{1}{7}}_{k_7}\vec{r}_7$$

(iii) $$\vec{x} = \frac{1}{4}\vec{c}_1 + \frac{1}{4}\vec{c}_2 + \frac{1}{2}\vec{c}_3$$

Here $\vec{x}$ is a weighted average of each test for a given student where Test 1 is 25% of grade, Test 2 is 25% of grade, and Test 3 is 50% of grade

Problem 2

(i) $$\vec{c}_1 = ((408, 762), (144, 620), \ldots)$$

$$\vec{c}_2 = ((847, 170), (479, 211), \ldots)$$

$$\vdots$$

or as matrices

$$\vec{c}_1 = \begin{pmatrix} 408,762 \\ 144,620 \\ \vdots \end{pmatrix} \quad \vec{c}_2 = \begin{pmatrix} 847,170 \\ 479,211 \end{pmatrix} \ldots$$

$$x_1 = k_1\ 408,762 + k_2\ 847,170 + \cdots$$

If X_1 is the average population of Philadelphia county over four years

then $X_1 = \underbrace{\frac{1}{4}}_{k_1} 408,762 + \underbrace{\frac{1}{4}}_{k_2} 847,170$

$+ \underbrace{\frac{1}{4}}_{k_3} (1,552,572) +$

$\underbrace{\frac{1}{4}}_{k_4} (1,436,287)$

or $\vec{X} = \frac{1}{4}\vec{C}_1 + \frac{1}{4}\vec{C}_2 + \frac{1}{4}\vec{C}_3 + \frac{1}{4}\vec{C}_4$

(iii) For example

$\vec{r}_1 = ((408,762), (847,170), (1,552,572), \cdots)$

$\vec{r}_2 = ((144,620), (479,211), (556,279), \cdots)$

or as matrices

$$\vec{r}_1 = \begin{pmatrix} 408,762 \\ 847,170 \\ \vdots \end{pmatrix} \quad \vec{r}_2 \begin{pmatrix} 144,620 \\ 479,211 \\ \vdots \end{pmatrix} \cdots$$

$X_1 = k_1\, 408,762 + k_2 (144,620) + \cdots$

$k_7 (16,810)$

If x_1 is the average population of the five counties in each sampled year

then $$x_1 = \frac{1}{5}(408,762) + \frac{1}{5}(144,620) + \frac{1}{5}(414,234) + \frac{1}{5}(44,197) + \frac{1}{5}(16,810)$$

so

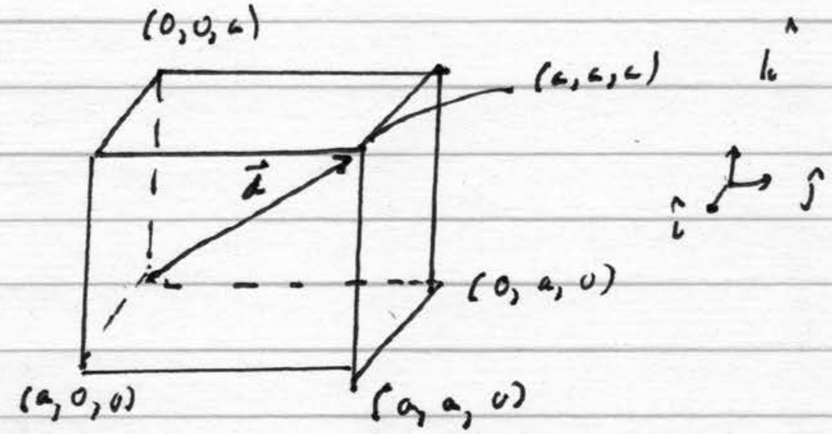
$$\vec{x} = \frac{1}{5}\vec{r}_1 + \frac{1}{5}\vec{r}_2 + \frac{1}{5}\vec{r}_3 + \frac{1}{5}\vec{r}_4 + \frac{1}{5}\vec{r}_5$$

(iii) $$\vec{x} = \frac{\vec{r}_1}{3} + \frac{\vec{r}_2}{3} + \frac{\vec{r}_3}{3}$$

is the average population of Philadelphia, Bucks, and Delaware counties in each sampled year

Problem 3

Find the angle θ between a diagonal of a cube and one of its edges.



Let

$$\vec{d} = (a, a, a)$$

and the vectors

$$\vec{V}_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \qquad \vec{V}_3 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\vec{V}_1$ and $\vec{d}$

$$\cos\theta = \frac{\vec{V}_1 \cdot \vec{d}}{\|\vec{V}_1\| \, \|\vec{d}\|} = \frac{a^2}{a\sqrt{3}\, a} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.7^\circ$$

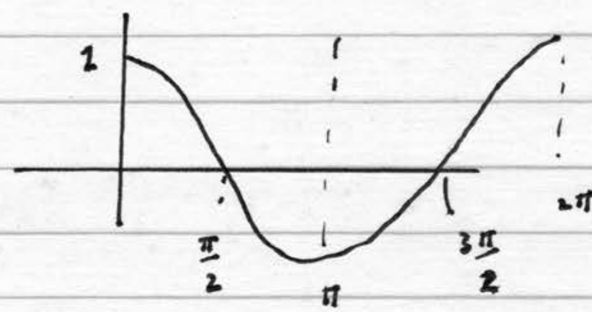
Problem 4

What can you say about the angle between the nonzero vectors $\vec{U}$ and $\vec{v}$ in $\mathbb{R}^2$ if

$$\vec{U} \cdot \vec{v} > 0 \ ?$$

$$\vec{U} \cdot \vec{v} = \|\vec{U}\| \, \|\vec{v}\| \cos\theta$$

If $\vec{U} \cdot \vec{v} > 0$ then $0 < \cos\theta < 1$



$0 < \theta < \pi/2$

or $3\pi/2 < \theta < 2\pi$

If $\vec{U} \cdot \vec{v} < 0$ then $\cos\theta < 0$

or

$\pi/2 < \theta < 3\pi/2$

Problem 5

Let us first explore the example of the Calculus book by Howard Anton

ISBN 0-471-15307-9

Check Digit 9

$\vec{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$

$\vec{b} = (0, 4, 7, 1, 1, 5, 3, 0, 7)$

$\vec{a} \cdot \vec{b} = 0 + 8 + 21 + 4 + 5 + 30 + 21 + 0 + 63 = 152$

$$\frac{\vec{a} \cdot \vec{b}}{11} = \frac{152}{11} = 13 \ {}^{9}/_{11}$$

remainder → 9

so check digit is 9

Now verify

1-56592-170-7

is a valid ISBN by computing its check digit

$\vec{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$

$\vec{b} = (1, 5, 6, 5, 9, 2, 1, 7, 0)$

$\vec{a} \cdot \vec{b} = 1 \cdot 1 + 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 5 + 5 \cdot 9 + 6 \cdot 2 + 7 \cdot 1 + 8 \cdot 7 + 9 \cdot 0$

$\vec{a} \cdot \vec{b} = 1\cdot1 + 2\cdot5 + 3\cdot6 + 4\cdot5 + 5\cdot9$

$+ 6\cdot2 + 7\cdot1 + 8\cdot7 + 9\cdot0$

$\vec{a}\cdot\vec{b} = 1 + 10 + 18 + 20 + 45 +$

$12 + 7 + 56 + 0$

$\vec{a}\cdot\vec{b} = 30 + 19 + 45 + 19 + 56$

$\vec{a}\cdot\vec{b} = 49 + 75 + 45 = 120 + 49 =$

169

$\frac{\vec{a}\cdot\vec{b}}{11} = \frac{169}{11} = 15 \; 4/11$

remainder $\rightarrow$ 4

Check number is $4 \neq 7$ so

1-56592-170-7 is <u>not</u>

a valid ISBN!

Problem 6

The intersection of two planes through

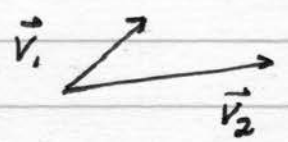
$$(0,0,0)$$

is probably a line, but it could be a plane (if the two planes are identical).
It can not be a point.

Problem 7

What geometric property must a set of two vectors have if they are to span $\mathbb{R}^2$?

These two vectors must be linearly independent!



$$\vec{V}_1 \neq \alpha \vec{V}_2$$

For

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 = \tilde{O}$$

both c_1, c_2 must vanish

If these two vectors are linearly dependent

$$\vec{V}_1 = \alpha \vec{V}_2$$

they only span $\mathbb{R}^1$

Problem 8

What geometric property must a set of three vectors in $\mathbb{R}^3$ have if they are to span $\mathbb{R}^3$?

The three vectors must be linearly independent. All three vectors can not lie in the plane; otherwise, they span $\mathbb{R}^2$.

Problem 9

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

If $\quad (\vec{w} - c\vec{v}) \cdot \vec{v} = 0$

then $\quad \vec{w} \cdot \vec{v} - c\,\vec{v} \cdot \vec{v} = 0$

or

$$c = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

$$\vec{w} \cdot \vec{v} = \vec{w}^{T} \vec{v} = (1 \;\; 5) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6$$

$$\vec{v} \cdot \vec{v} = \vec{v}^{T} \vec{v} = (1 \;\; 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

$$c = \frac{6}{2} = 3$$

Proof

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad c\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\vec{w} - c\vec{v} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$(\vec{w} - c\vec{v}) \cdot \vec{v} = (-2 \;\; 2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2 + 2 = 0$$

Problem 10.

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\|\vec{v}\| = (\vec{v}\cdot\vec{v})^{1/2} = (\vec{v}^{\,T}\vec{v})^{1/2} = \left[\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right]^{1/2} =$$

$$(1+1\cdots+1)^{1/2} = (9)^{1/2}$$

$$= 3$$

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Check $\hat{u}\cdot\hat{u} = \frac{1}{9}\,9 = 1$ or $\|\hat{u}\| = 1$

Let us find $\vec{w}$, where

$$\vec{W} \cdot \vec{V} = 0 \quad \text{or} \quad \vec{w}^{T} \vec{V} = 0$$

$$\vec{W} = (w_1, w_2, w_3, w_4, W_5, W_6, w_7, W_8, W_9)$$

$$\vec{W} = (w_1 \; w_2 \; w_3 \; w_4 \; W_5 \; W_6 \; W_7 \; W_8 \; W_9)$$

$$\vec{W} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \\ W_8 \\ W_9 \end{pmatrix} \qquad \vec{V} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{W}^{T} \vec{V} = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 = 0$$

For example $W_1 = W_2 = W_3 = W_4 = \quad = -1$

$W_5 = W_6 = W_7 = W_8 = +1$

$W_9 = 0$

$$\vec{W} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \|\vec{W}\| = \left(\vec{W}^{T} \vec{W}\right)^{1/2}$$

$$\vec{w}^T\vec{w} \quad (-1 \ -1 \ -1 \ -1 \ +1 \ +1 \ +1 \ +1 \ 0)\begin{pmatrix}-1\\-1\\-1\\-1\\1\\1\\1\\1\\0\end{pmatrix} =$$

$$1+1+1+1+1+1+1+1+0$$

$$= 8$$

$$\boxed{\|\vec{w}\| = \sqrt{8}}$$

$$\hat{w} = \frac{\vec{w}}{\sqrt{8}} = \frac{1}{\sqrt{8}}\begin{pmatrix}-1\\-1\\-1\\-1\\1\\1\\1\\1\\0\end{pmatrix}$$

N. B.
$\hat{w}\cdot\hat{w} = 1$
or $\|\hat{w}\| = 1$

Note
$$\hat{w}\cdot\vec{v} = \hat{w}^T\vec{v} = \frac{1}{\sqrt{8}}(-1 \ -1 \ -1 \ -1 \ 1 \ 1 \ 1 \ 1 \ 0)\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{8}}\left(-1 -1 -1 -1 +1 +1 +1 +0\right)$$

$$= 0$$